SOLVED PROBLEMS ON LAPLACE TRANSFORM

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November 2006

LAPLACE TRANSFORM

Many mathematical problems are solved using transformations. The idea is to transform the problem into another problem that is easier to solve. Once a solution is obtained, the inverse transform is used to obtain the solution to the original problem. The Laplace transform is an important tool that makes solution of linear constant coefficient differential equations much easier. The Laplace transform transforms the differential equations into algebraic equations which are easier to manipulate and solve. Once the solution is obtained in the Laplace transform domain is obtained, the inverse transform is used to obtain the solution to the differential equation. Laplace transform is an essential tool for the study of linear time-invariant systems.

In this handout a collection of solved examples and exercises are provided. They are grouped into two parts: background material and Laplace transform. The background material is supposed to be covered in the prerequisite materials namely MATH 001, MATH002, MATH 101,MATH 102, MATH 201 and MATH 260. Most of the problems in this part are easy. They are listed here for quick review and to ensure that students can perfectly solve these problems.

The second part is directly related to the material covered in this course. Read the text book for more details. The intension here is not to replace the textbook but rather to give supplementary material to improve the student understanding.

LEARNING OBJECTIVES

A list of learning objectives that students are expected acquire in the course is listed below. This problem set is not covering all of them. Read the textbook and make sure that you know all the items below.

1. To define (mathematically) the unit step and unit impulse.
2. To express some simple functions in terms of unit step and/or unit impulse
3. To state and use the sampling property of the impulse.
4. To determine if a function is of exponential order or not.
5. To know basic integration rules (including integration by parts)
6. To be able to factor second order polynomials
7. To perform algebraic manipulation of complex numbers.
8. To state the definition of Laplace transform.
9. To give sufficient conditions for existence of Laplace transform.
10. To obtain Laplace transform of simple functions (step, impulse, ramp, pulse, sin, cos, …)
11. To obtain Laplace transform of functions expressed in graphical form.
12. To know the linear property of Laplace transform.
13. To know Laplace transform of integral and derivatives (first and high orders derivatives.
14. To obtain inverse Laplace transform of simple function using the Table of Laplace transform pairs.
15. To use the method of partial fraction expansion to express strictly proper functions as the sum of simple factors (for the cases: simple poles, complex poles and repeated poles).
16. To perform long division and know the reason for using it in inverse Laplace transform.
17. To obtain inverse Laplace transform.
18. To solve constant coefficient linear ordinary differential equations using Laplace transform.
19. To derive the Laplace transform of time-delayed functions.
20. To know initial-value theorem and how it can be used.
21. To know final-value theorem and the condition under which it can be used.

BACKGROUND MATERIAL

Topics:

1. Factoring a Second Order Polynomial
2. Integration
3. Limits
4. Complex Number Manipulation
5. Long division
6. Useful identities

1. FACTORING A SECOND ORDER POLYNOMIAL

Factoring of polynomials are needed in solving problems related to Laplace Transform. Factoring high order polynomials may not be easy and computers may be needed to do the factorization. Factoring second order polynomials can be done manually.

EXAMPLE 1:

Factor the second order polynomial $P(x) = x^2 + 5x + 6$

Answer:

This is a simple problem. Find two numbers such that their product is six and their sum is five. They are two and three. $P(x) = x^2 + 5x + 6 = (x+2)(x+3)$

EXAMPLE 2:

Factor the second order polynomial $P(x) = x^2 + 0.9x - 2.52$

Answer:

It is not easy to guess two numbers such that their product is -2.52 and their sum is 0.9. We can use the formula to find the roots then write the factors.

$$roots\ of\ \ ax^2 + bx + c = 0\ \ are\ \frac{-b \pm \sqrt{b^2 - 4(a)(c)}}{2(a)}$$

$$roots\ of\ \ x^2 + 0.9x - 2.25 = 0\ \ are\ \frac{-0.9 \pm \sqrt{0.9^2 - 4(1)(-2.52)}}{2(1)} = 1.2\ and\ \ -2.1$$

$$P(x) = x^2 + 0.9x - 2.25 = (x+2.1)(x-1.2)$$

EXAMPLE 3:

Factor the second order polynomial $P(x) = x^2 + 4x + 13$

Answer:

We can use the formula to find the roots then write the factors.

$$roots\ of\ \ x^2 + 4x + 13 = 0\ \ are\ \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)} = -2 \pm j3$$

$$P(x) = x^2 + 4x + 13 = (x + 2 + j3)(x + 2 - j3)$$

EXAMPLE 4:

Express the second order polynomial $P(x) = x^2 + 4x + 13$ in the sum of square form

Answer:

$$P(s) = x^2 + 4x + 13 = \left[x^2 + 4x + \left(\frac{4}{2}\right)^2\right] - \left(\frac{4}{2}\right)^2 + 13 = \left[\left(x + \frac{4}{2}\right)^2\right] - \left(\frac{4}{2}\right)^2 + 13 = (x+2)^2 + (3)^2$$

EXAMPLE 5:

Factor the second order polynomial $P(x) = 2x^2 + 4x - 6$

Answer:

$P(x) = 2x^2 + 4x - 6 = 2\left(x^2 + 2x - 3\right) = 2\left(x + 3\right)(x - 1)$

EXAMPLE 6:

Factor the second order polynomial $P(x) = (x + 2)^2 + 16$

Answer:

This polynomial is expressed as the sum of squares. It has two complex poles. The real part of the roots is -2 and imaginary parts are ± 4.

$P(x) = (x + 2)^2 + 16 = (x + 2 + j4)(x + 2 - j4)$

Exercise 1: Factor $P(x) = x^2 + 12x + 32$

Exercise 2: Factor $P(x) = x^2 + 4.4x + 1.9$

Exercise 3: Factor $P(x) = 2x^2 + 2x + 11$

Exercise 4: Factor $P(x) = -x^2 + 2x + 13$

Exercise 5: Express $P(x) = x^2 + 2x + 17$ in sum of squares form.

INTEGRATION

Integration is a large topic. Here we will concentrates on integrals often used in problems related to Laplace transform. In particular, we concentrate on integrals involving e^{ax}.

EXAMPLE 1:

Evaluate the integral $\int_0^2 e^{2x}dx$

Answer:

$$\int_0^2 e^{2x}dx = \left.\frac{e^{2x}}{2}\right|_0^2 = \frac{e^4}{2} - \frac{e^0}{2} = \frac{e^4}{2} - \frac{1}{2} = \frac{1}{2}\left(e^4 - 1\right)$$

EXAMPLE 2:

Evaluate the integral $\int_a^b e^{cx}dx$ where a, b and c are constants and c is not zero.

Answer:

$$\int_a^b e^{cx}dx = \left.\frac{e^{cx}}{c}\right|_a^b = \frac{1}{c}\left(e^{cb} - e^{ca}\right)$$

EXAMPLE 3:

Evaluate the integral $\int_0^\infty e^{-cx}dx$ where c is a positive constant.

Answer:

$$\int_0^\infty e^{-cx}dx = \left.\frac{e^{-cx}}{-c}\right|_0^\infty = \frac{1}{-c}\left(e^{-c\infty} - e^{-c0}\right) = \frac{1}{-c}(0-1) = \frac{1}{c}$$

EXAMPLE 4:

Evaluate the integral $\int_5^\infty e^{-2t}dt$.

Answer:

$$\int_5^\infty e^{-2t}dt = \left.\frac{e^{-2t}}{-2}\right|_5^\infty = \frac{1}{-2}\left(e^{-2(\infty)} - e^{-2(5)}\right) = \frac{1}{-2}\left(0 - e^{-10}\right) = \frac{e^{-10}}{2}$$

EXAMPLE 5:

Evaluate the integral $\int_1^3 te^{-2t}dt$.

Answer:

Using integration by parts

$$define\ \ u = t \quad \Rightarrow \qquad du = dt$$

$$dv = e^{-2t}dt \quad \Rightarrow \quad v = \frac{e^{-2t}}{-2}$$

$$\int_1^3 te^{-2t}dt = vu \Big|_1^3 - \int_1^3 vdu = \frac{e^{-2t}}{-2}t\Big|_1^3 - \int_1^3 \frac{e^{-2t}}{-2}dt = \frac{3e^{-6}}{-2} - \frac{e^{-2}}{-2} - (............)$$

EXAMPLE 6:

Evaluate the integral $\int_1^3 t^2e^{-2t}dt$.

Answer:

EXAMPLE 7:

Evaluate the integral $\int_1^3 \sin(5t)e^{-2t}dt$.

Answer:

EXAMPLE 8:

Evaluate the integral $\int_0^\infty \sin(5t)e^{-ct}dt$ where c is a positive constant.

Answer:

EXAMPLE 9:

Evaluate the integral $\int_0^{10} f(t)dt$ where $f(t) = \begin{cases} 2t & 0 \le t \le 2 \\ 2t+5 & 2 < t \le 10 \end{cases}$

Answer:

$$\int_0^{10} f(t)dt = \int_0^2 f(t)dt + \int_2^{10} f(t)dt = \int_0^2 2tdt + \int_2^{10}(2t+5)dt = \frac{2t^2}{2}\Big|_0^2 + \frac{2t^2}{2} + 5t\Big|_2^{10} = 44$$

LIMITS

EXAMPLE 1:

Evaluate the limit $\lim_{t\to\infty}\frac{t^2+3t}{t^2-1}$.

Answer:

$\lim_{t\to\infty}\frac{t^2+3t}{t^2-1}=\frac{\infty}{\infty}$ Dividing by the highest power and taking the limit

$$\lim_{t\to\infty}\frac{t^2+3t}{t^2-1}=\lim_{t\to\infty}\frac{\frac{t^2}{t^2}+\frac{3t}{t^2}}{\frac{t^2}{t^2}-\frac{1}{t^2}}=\lim_{t\to\infty}\frac{1+\frac{3}{t}}{1-\frac{1}{t^2}}=\frac{1+0}{1-0}=1$$

EXAMPLE 2:

Evaluate the limit $\lim_{t\to\infty}\sin(10t)e^{-2t}$.

Answer:

$\lim_{t\to\infty}\sin(10t)e^{-2t}=0$

EXAMPLE 3:

Evaluate the limit $\lim_{t\to\infty}t\ e^{-2t}$.

Answer:

$\lim_{t\to\infty}t\ e^{-2t}=0$

EXAMPLE 4:

Evaluate the limit $\lim_{t\to\infty}t\ e^{3t}$.

Answer:

$\lim_{t\to\infty}t\ e^{3t}=\infty$

EXAMPLE 5:

Evaluate the limit $\lim_{t\to 0}\frac{t^2+3t}{t^2-1}$.

Answer:

$\lim_{t\to 0}\frac{t^2+3t}{t^2-1}=\frac{0}{-1}=0$

EXAMPLE 6:

Evaluate the limit $\lim_{t \to 0} \frac{t^2 + 3t}{t^2 - t}$.

Answer:

$\lim_{t \to 0} \frac{t^2 + 3t}{t^2 - t} = \frac{0}{0}$ using L'Hopital's rule

$\lim_{t \to 0} \frac{t^2 + 3t}{t^2 - t} = \lim_{t \to 0} \frac{2t + 3}{2t - 1} = \frac{3}{-1} = -3$

COMPLEX NUMBER MANIPULATION

EXAMPLE 1:

Evaluate $(2+j3)+(-4+j0.3)$.

Answer:

$(2+j3)+(-4+j0.3)=-2+j3.3$

EXAMPLE 2:

Evaluate $(2+j3)(-4+j0.3)$.

Answer:

$(2+j3)(-4+j0.3)=-8.9-j11.4$

EXAMPLE 3:

Evaluate $\dfrac{2+j3}{4-j2}$.

Answer:

$$\frac{2+j3}{4-j2}=0.1+j0.8$$

EXAMPLE 4:

Let $F(s)=\dfrac{s+1}{s-2}$. Evaluate $F(2+j3)$.

Answer:

$$F(2+j3)=\frac{2+j3+1}{2+j3-2}=1-j$$

EXAMPLE 5:

Let $F(s)=\dfrac{s+3}{(s+2)(s+1)}$. Evaluate $F(2+j2)$

Answer:

$$F(2+j2)=\frac{2+j2+3}{(2+j2+2)(2+j2+1)}=0.2615\text{ - j}\,0.2077$$

LONG DIVISION

EXAMPLE 1:

Let $F(s) = \dfrac{s^2 + 9s + 3}{(s+2)(s+1)}$. Perform long division and determine the quotient and the remainder.

Answer:

$$
\begin{array}{r|l}
 & \quad 1 \\
\hline
s^2 + 3s + 2 & \quad s^2 + 9s + 3 \\
 & -s^2 - 3s - 2 \\
\cline{2-2}
 & \qquad 6s + 1
\end{array}
$$

$$F(s) = \frac{s^2 + 9s + 3}{(s+2)(s+1)} = \frac{s^2 + 9s + 3}{s^2 + 3s + 2} = 1 + \frac{6s + 1}{s^2 + 3s + 2}$$

EXAMPLE 2:

Let $F(s) = \dfrac{3s^3 + 17s^2 + 33s + 15}{s^3 + 6s^2 + 11s + 6}$. Perform long division and determine the quotient and the remainder.

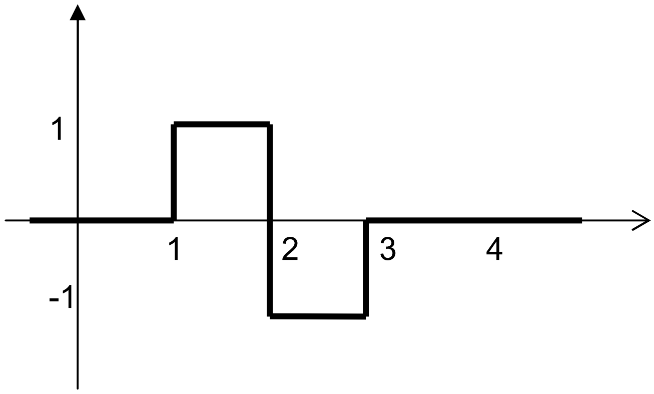
Answer:

$$F(s) = 3 + \frac{-s^2 - 3}{s^3 + 6s^2 + 11s + 6}$$

MISCELLANEOUS PROBLEMS

EXAMPLE 1:

Express the following function as a sum of shifted steps.

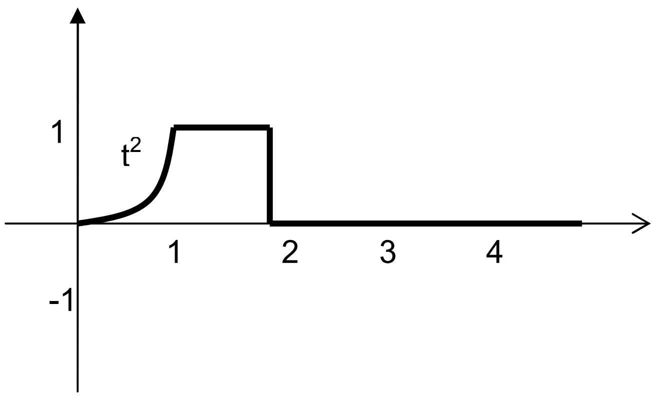


Answer:

$$f(t) = u(t-1) - 2u(t-2) + u(t-1)$$

EXAMPLE 2:

Express the following function in terms of shifted steps.



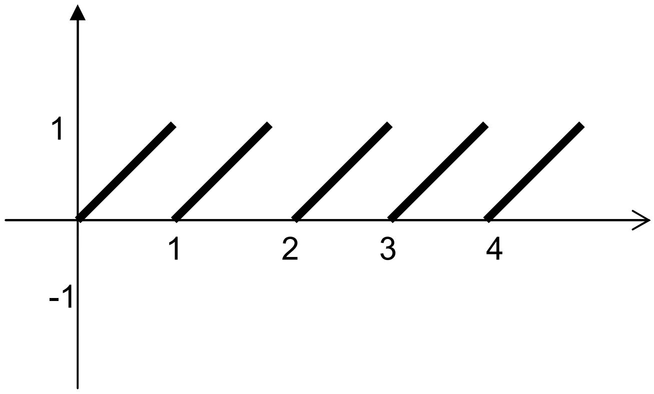
Answer:

$$f(t) = \begin{cases} t^2 & t \in [0,1] \\ 1 & t \in [1,2] \\ 0 & otherwise \end{cases}$$

$$f(t) = t^2(u(t) - t(t-1) + 1(u(t-1) - u(t-2))$$

EXAMPLE 3:

Obtain an analytical expression of the saw function.

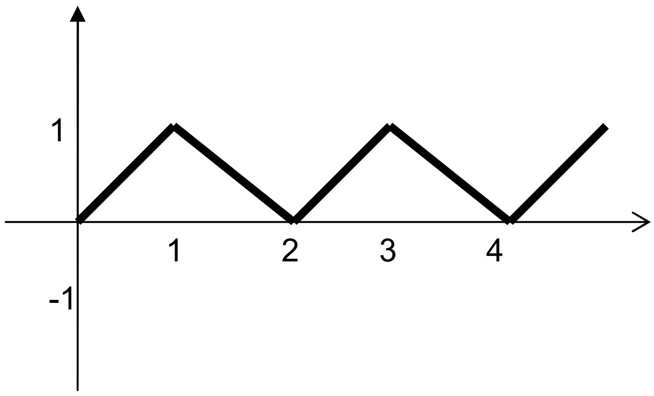


Answer:

$$f(t)=\begin{cases} t & t\in[0,1) \\ t-n & t\in[n,n+1) \\ 0 & t<0 \end{cases}$$

EXAMPLE 3:

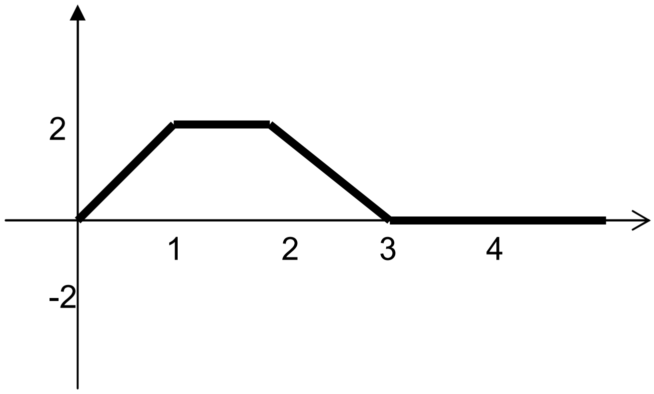
Obtain an analytical expression of the function.



Answer:

EXAMPLE 4:

Obtain an analytical expression of the function.



Answer:

$$f(t) = \begin{cases} 2t & t \in [0,1) \\ 2 & t \in [1,2) \\ -2t + 6 & t \in [2,3) \\ 0 & otherwise \end{cases}$$

USEFUL IDENTITIES

Often some problem can be made considerably easier if some identities are used.

useful identities

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

EXAMPLE 1:

LAPLACE TRANSFORM

Computing the Laplace transform using definition
Using Laplace transform Properties
Inverse Laplace transform

DEFINITION OF THE LAPLACE TRANSFORM

The Laplace transform converts a function of real variable f(t) into a function of complex variable F(s). The Laplace transform is defined[1] as

$$F(s) = L\{f(t)\} = \int_{0-}^{\infty} f(t)e^{-at}\ dt$$

The variable s is a complex variable that is commonly known as the Laplace operator.

EXAMPLE 1:

Find the Laplace transform of $f(t) = \begin{cases} 2 & t \in [0,2] \\ 0 & otherwise \end{cases}$

Answer:

$$F(s) = L\{f(t)\} = \int_{0-}^{\infty} f(t)e^{-at}\ dt = \int_{0}^{2} 2e^{-at}\ dt + \int_{2}^{\infty} 0e^{-at}\ dt = \frac{2}{s} - \frac{2}{s}e^{-2s}$$

EXAMPLE 2:

Find the Laplace transform of $f(t) = 1 - 2e^{-2t}$

Answer:

$$F(s) = L\{f(t)\} = \int_{0-}^{\infty} f(t)e^{-at}\ dt = \int_{0}^{\infty} (1 - 2e^{-2t})e^{-at}\ dt = \int_{0}^{\infty} e^{-at}\ dt - 2\int_{0}^{\infty} e^{-2t}e^{-at}\ dt = \frac{1}{s} - \frac{2}{s+2}$$

USING LAPLACE TRANSFORM PROPERTIES

Often it is possible to obtain Laplace Transforms of complicated functions using one or more of the Laplace transform properties. Some other problems can be made easy by applying trigonometric identities. In this section several examples are shown.

EXAMPLE 1:

Find the Laplace transform of $f(t) = t^2 e^{-2x} \cos(3t)$

Answer:

For convenience, let us define the following functions

$$g(t) = \cos(3t)$$

$$h(t) = e^{-2x} \cos(3t) = e^{-2x} g(t)$$

$$then\ f(t) = t^2\, h(t)$$

$$Let\ G(s) = L\{g(t)\}, H(s) = L\{h(t)\}, F(s) = L\{f(t)\}$$

$$G(s) = \frac{s}{s^2 + 9} \quad fromTable$$

$$H(s) = \frac{s+2}{(s+2)^2 + 9}$$

$$F(s) = -\frac{d}{ds}\left[-\frac{d}{ds} H(s)\right] = \frac{2(\mathrm{s}+2)(\mathrm{s}^2 + 4\mathrm{s} - 23)}{(\mathrm{s}^2 + 4\mathrm{s} + 13)^3}$$

EXAMPLE 2:

Find the Laplace transform of $f(t) = [\cos(3t)]^2$

Answer:

Using the trigonometric identity

$$\cos^2(3t) = \frac{1}{2}\left(1 + \cos(6t)\right)$$

$$L\{\cos^2(3t)\} = \frac{1}{2s} + \frac{s}{2(s^2 + 36)}$$

EXAMPLE 3:

Find the Laplace transform of $f(t) = \left(e^{-2t} - 1\right)^2$

Answer:

Using the trigonometric identity

$$(e^{-2t}-1)^2 = e^{-4t} - 2e^{-2t} + 1$$

$$L\{(e^{-2t}-1)^2\} = \frac{1}{s+4} - \frac{2}{s+2} + \frac{1}{s}$$

EXAMPLE 4:

Find the Laplace transform of $f(t) = \sin(2t)\cos(2t)$

Answer:

Using the trigonometric identity

$$\sin(2t)\cos(2t) = \frac{1}{2}\left(\sin(4t)\right)$$

$$L\{\sin(2t)\cos(2t)\} = \frac{4}{2(s^2+16)}$$

EXAMPLE 5:

Find the Laplace transform of $f(t) = t^2$

Answer:

Using the multiplication by t property

$$L\{t\} = \frac{1}{s^2}$$

$$L\{t^2\} = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

Exercises:

Find the Laplace transform of

1. $f(t) = \left[\cos(3t)\right]^3$
2. $f(t) = \sin^2(3t)$
3. $f(t) = \sin(3t-4)$

INVERSE LAPLACE TRANSFORM

The formula for computing the Laplace transform is well known. A simple way to compute the inverse transform is split the function as the sum of first and second order terms then obtain the inverse for each term alone and finally add them together. Standard Laplace transform tables can be used to obtain inverse transforms for simple terms.

EXAMPLE 1:

Find the inverse Laplace transform of $F(s) = \dfrac{2}{s+k}$

Answer:

$$f(t) = L^{-1}\left\{F(s)\right\} = 2e^{-kt}$$

EXAMPLE 2:

Find the inverse Laplace transform of $F(s) = \dfrac{2}{s^2+3s+2}$

Answer:

A first step in the solution is to factor the denominator

$$s^2 + 3s + 2 = (s+2)(s+1)$$

Then F(s) can be expressed in terms of the unknown parameters A and B

$$F(s) = \frac{2}{s^2+3s+2} = \frac{A}{s+2} + \frac{B}{s+1}$$

Use the formula to determine A and B

$$A = (s+2)F(s)\big|_{s=-2} = \left.\frac{2}{s+1}\right|_{s=-2} = -2$$

$$B = (s+1)F(s)\big|_{s=-1} = \left.\frac{2}{s+2}\right|_{s=-1} = 2$$

$$F(s) = \frac{-2}{s+2} + \frac{2}{s+1}$$

$$f(t) = L^{-1}\left\{F(s)\right\} = -2e^{-2t} + 2e^{-t} \qquad for \quad t \geq 0$$

EXAMPLE 3:

Find the inverse Laplace transform of $F(s) = \frac{2}{s+c} e^{-bs}$

Answer:

$$Let\ G(s) = \frac{2}{s+c} \Rightarrow g(t) = L^{-1}\ \{G(s)\} = 2e^{-ct}$$
$$f(t) = L^{-1}\ \{G(s)e^{-bs}\} = 2e^{-k(t-b)}\ u(t-b)$$

EXAMPLE 4:

Find the inverse Laplace transform of $F(s) = \frac{1}{(s+2)^3}$

Answer:

$$f(t) = L^{-1}\ \{F(s)\} =$$

EXAMPLE 5:

Find the inverse Laplace transform of $F(s) = \frac{3s+5}{s^2+7}$

Answer:

$$f(t) = L^{-1}\ \{F(s)\} = L^{-1}\ \left\{\frac{3s}{s^2+7} + \frac{5}{s^2+7}\right\}$$
$$= L^{-1}\ \left\{3\ \frac{s}{s^2+\left(\sqrt{7}\right)^2} + \frac{5}{\sqrt{7}} \frac{\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2}\right\}$$
$$= 3\cos\left(\sqrt{7}\quad t\right) + \frac{5}{\sqrt{7}}\sin\left(\sqrt{7}\quad t\right)$$

EXAMPLE 7:

Find the inverse Laplace transform of $F(s) = \frac{5}{s^2-9}$

Answer:

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{5}{s^2 - 9}\right\} = L^{-1}\left\{\frac{A}{s-3} + \frac{B}{s+3}\right\}$$

$$A = (s-3)\left.\frac{5}{(s-3)(s+3)}\right|_{s=3} = \frac{5}{6}$$

$$B = (s+3)\left.\frac{5}{(s-3)(s+3)}\right|_{s=-3} = \frac{5}{-6}$$

$$= \frac{5}{6}e^{3t} - \frac{5}{6}e^{-3t}$$

EXAMPLE 8:

Find the inverse Laplace transform of $F(s) = \dfrac{s+1}{s^2(s+2)^3}$

Answer:

$$F(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+2)} + \frac{D}{(s+2)} + \frac{E}{(s+2)}$$

$$A = s^2 \left. F(s)\right|_{s=0} = \frac{1}{8}$$

$$B = \frac{d}{ds}\left.\left(s^2 \; F(s)\right)\right|_{s=0} = \frac{-1}{16}$$

$$C = (s+2)^3 \left. F(s)\right|_{s=-2} = \frac{1}{4}$$

$$D = \frac{d}{ds}\left.\left((s+2)^3 \; F(s)\right)\right|_{s=-2} = 0$$

$$E = \frac{1}{2!}\frac{d^2}{ds^2}\left.\left((s+2)^3 \; F(s)\right)\right|_{s=-2} = \frac{1}{16}$$

$$= \frac{-1}{16} + \frac{1}{8}t + \frac{1}{16}e^{-2t} \text{ ???????????}$$

EXAMPLE 9:

Find the inverse Laplace transform of $F(s) = \frac{3(s+3)}{s^2+6s+8}$

Answer:

$$f(t) = L^{-1}\left\{F(s)\right\} = 1.5e^{-4t} + 1.5e^{-2t}$$

EXAMPLE 10:

Find the inverse Laplace transform of $F(s) = \frac{1}{s(s^2+34.5s+1000)}$

Answer:

$$f(t) = L^{-1}\left\{F(s)\right\} = 0.001 + 0.00119e^{-17.25t}\sin(26.5t + 56.9)$$

EXAMPLE 12:

Find the inverse Laplace transform of $F(s) = \frac{e^{-3s}}{s(s^2+3s+2)}$

Answer:

$$Let G(s) = \frac{1}{s(s^2+3s+2)} \Rightarrow F(s) = G(s)e^{-3s}$$

$$g(t) = L^{-1}\left\{G(s)\right\} = L^{-1}\left\{\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}\right\}$$

$$A = 0.5, \quad B = 0.5, \; C = -1$$

$$g(t) = 0.5 + 0.5e^{-2t} - e^{-t}$$

$$f(t) = L^{-1}\left\{F(s)\right\} = \left\{0.5 + 0.5e^{-2(t-3)} - e^{-(t-3)}\right\}u(t-3)$$

FINAL VALUE THEOREM

The final value theorem can provide the steady state value of a time-domain function from its Laplace transform without going through the inverse Laplace transform procedure.

Conditions that must be satisfied:

- No poles on imaginary axis (except simple pole at origin)
- No poles on having positive real part

EXAMPLE 1:

What is the steady state value of f(t) if you know that

$$F(s) = \frac{2}{s(s+1)(s+2)(s+3)}$$

Answer:

A simple pole at origin and all other poles have negative real part.

$$f(\infty) = \lim_{s \to 0} s\,F(s) = \lim_{s \to 0} \frac{2}{(s+1)(s+2)(s+3)} = \frac{1}{3}$$

EXAMPLE 2:

What is the steady state value of f(t) if you know that $F(s) = \frac{1}{(s-1)(s+2)}$

Answer:

The final value theorem is not applicable because F(s) has a pole s = 1.

EXAMPLE 3:

What is the steady state value of f(t) if you know that $F(s) = \frac{1}{(s+2)^2(s+4)}$

Answer:

All poles have negative real part.

$$f(\infty) = \lim_{s \to 0} s\,F(s) = \lim_{s \to 0} \frac{s}{(s+2)^2(s+4)} = 0$$

EXAMPLE 4:

What is the steady state value of f(t) if you know that $F(s) = \frac{10}{s^2(s+1)}$

Answer:
The final value theorem is not applicable because F(s) has two poles at origin.

EXAMPLE 5:

What is the steady state value of f(t) if you know that $F(s) = \frac{10}{(s+1)(s^2+1)}$

Answer:
The final value theorem is not applicable because F(s) has two poles on the imaginary axis (j and –j).

EXAMPLE 6:

What is the steady state value of f(t) if you know that

$$F(s) = \frac{b}{s(s+1)(s+a)}, \; where \;\; a > 0$$

Answer:

$$f(\infty) = \lim_{s \to 0} s\,F(s) = \lim_{s \to 0} \frac{s\,b}{s(s+1)(s+a)} = \frac{b}{a},$$

SOLVING ODE

Solving ODE is one important application for Laplace transform. The general procedure is shown below.

1. Use Laplace transform to convert the ODE into algebraic equation.
2. Solve the algebraic equation for the unknown function
3. Use Partial fraction expansion to express the unknown function as the sum of first and second order terms
4. Use inverse Laplace transform to obtain the solution to the original problem

EXAMPLE 1:

$$\ddot{x}(t) + 4\dot{x}(t) + 3x(t) = 1$$

$$x(0) = 0; \dot{x}(0) = 0$$

Answer:

Step1: Convert to algebraic equation	$\ddot{x}(t) + 4\dot{x}(t) + 3x(t) = 1$ $\{s^2 X(s) - sx(0) - \dot{x}(0)\} + 4\{sX(s) - x(0)\} + 3X(s) = \frac{1}{s}$ *subistiuting initial conditions* $\{s^2 X(s)\} + 4\{sX(s)\} + 3X(s) = \frac{1}{s}$
Step2: Solve for X(s)	$\{s^2 X(s)\} + 4\{sX(s)\} + 3X(s) = \frac{1}{s}$ $(s^2 + 4s + 3)X(s) = \frac{1}{s}$ $X(s) = \frac{1}{s(s^2 + 4s + 3)}$
Step3: Use partial fraction expansion	$X(s) = \frac{1}{s(s^2 + 4s + 3)} = \frac{A}{s} + \frac{B}{(s+3)} + \frac{C}{(s+1)}$ $A = (s) \left. \frac{1}{s(s^2 + 4s + 3)} \right\|_{s=0} = \left. \frac{1}{(s^2 + 4s + 3)} \right\|_{s=0} = \frac{1}{3}$ $B = (s+3) \left. \frac{1}{s(s^2 + 4s + 3)} \right\|_{s=-3} = \left. \frac{1}{s(s+1)} \right\|_{s=-3} = \frac{1}{-3(-2)} = \frac{1}{6}$ $C = (s+1) \left. \frac{1}{s(s^2 + 4s + 3)} \right\|_{s=-1} = \left. \frac{1}{s(s+3)} \right\|_{s=-1} = \frac{1}{-1(2)} = -\frac{1}{2}$
Step4: Inverse Laplace Transform	$X(s) = \frac{1}{s(s^2 + 4s + 3)} = \frac{\frac{1}{3}}{s} + \frac{\frac{1}{6}}{(s+3)} + \frac{-\frac{1}{2}}{(s+1)}$ $x(t) = \frac{1}{3} + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t} \quad for\ t \geq 0$

EXAMPLE 2:

$\ddot{x}(t)+4\dot{x}(t)+3x(t)=1$

$x(0)=1; \dot{x}(0)=2$

Answer:

Step1: Convert to algebraic equation	$\ddot{x}(t)+4\dot{x}(t)+3x(t)=1$ $\{s^2X(s)-sx(0)-\dot{x}(0)\}+4\{sX(s)-x(0)\}+3X(s)=\frac{1}{s}$ *subistiuting initial conditions* $\{s^2X(s)-s-2\}+4\{sX(s)-1\}+3X(s)=\frac{1}{s}$
Step2: Solve for X(s)	$\{s^2X(s)\}+4\{sX(s)\}+3X(s)=\frac{1}{s}$ $(s^2+4s+3)X(s)=\frac{1}{s}+s+3=\frac{1+s^2+3s}{s}$ $X(s)=\frac{s^2+3s+1}{s(s^2+4s+3)}$
Step3: Use partial fraction expansion	$X(s)=\frac{1}{s(s^2+4s+3)}=\frac{A}{s}+\frac{B}{(s+3)}+\frac{C}{(s+1)}$ $A=(s)\left.\frac{s^2+3s+1}{s(s^2+4s+3)}\right\|_{s=0}=\left.\frac{s^2+3s+1}{(s^2+4s+3)}\right\|_{s=0}=\frac{1}{3}$ $B=(s+3)\left.\frac{s^2+3s+1}{s(s^2+4s+3)}\right\|_{s=-3}=\left.\frac{s^2+3s+1}{s(s+1)}\right\|_{s=-3}=\frac{1}{-3(-2)}=\frac{1}{6}$ $C=(s+1)\left.\frac{s^2+3s+1}{s(s^2+4s+3)}\right\|_{s=-1}=\left.\frac{s^2+3s+1}{s(s+3)}\right\|_{s=-1}=\frac{-1}{-1(2)}=\frac{1}{2}$
Step4: Inverse Laplace Transform	$X(s)=\frac{1}{s(s^2+4s+3)}=\frac{\frac{1}{3}}{s}+\frac{\frac{1}{6}}{(s+3)}+\frac{\frac{1}{2}}{(s+1)}$ $x(t)=\frac{1}{3}+\frac{1}{6}e^{-3t}+\frac{1}{2}e^{-t} \quad \text{for } t\geq 0$

EXAMPLE 3:

$$\ddot{x}(t) + 4x(t) = 1$$

$$x(0) = 0; \dot{x}(0) = 0$$

Answer:

Step1: Convert to algebraic equation	$\ddot{x}(t) + 4x(t) = 1$ $\left\{s^2 X(s) - sx(0) - \dot{x}(0)\right\} + 4X(s) = \dfrac{1}{s}$ *subistiuting initial conditions* $\left\{s^2 X(s)\right\} + 4X(s) = \dfrac{1}{s}$
Step2: Solve for X(s)	$\left\{s^2 X(s)\right\} + 4X(s) = \dfrac{1}{s}$ $\left(s^2 + 4\right) X(s) = \dfrac{1}{s}$ $X(s) = \dfrac{1}{s\left(s^2 + 4\right)}$
Step3: Use partial fraction expansion	$X(s) = \dfrac{1}{s\left(s^2 + 4\right)} = \dfrac{A}{s} + \dfrac{Bs + C}{\left(s^2 + 4\right)}$ $A = s \left. \dfrac{1}{s\left(s^2 + 4\right)} \right\vert_{s=0} = \left. \dfrac{1}{\left(s^2 + 4\right)} \right\vert_{s=0} = \dfrac{1}{4}$ $(Bs + c)s + A\left(s^2 + 4\right) = 2$ $\Rightarrow \quad A = -B, \; C = 0$
Step4: Inverse Laplace Transform	$X(s) = \dfrac{1}{s\left(s^2 + 4\right)} = \dfrac{0.25}{s} + \dfrac{-0.25s}{\left(s^2 + 4\right)}$ $x(t) = 0.25 - 0.25\cos(2t) \qquad \text{for } t \geq 0$