Examples for Runge-Kutta methods

We will solve the initial value problem,

$$\frac{du}{dx} = -2u + x + 4 \quad , \quad u(0) = 1 \, ,$$

to obtain u(0.2) using $\Delta x = 0.2$ (i.e., we will march forward by just one Δx).

(i) <u>3rd order Runge-Kutta method</u>

For a general ODE, $\frac{du}{dx} = f(x, u(x))$, the formula reads

$$u(x+\Delta x) = u(x) + (1/6)\,(K_1 + 4\,K_2 + K_3)\,\Delta x \, ,$$
$$K_1 = f(x, u(x)) \, ,$$
$$K_2 = f(x+\Delta x/2, u(x)+K_1\Delta x/2) \, ,$$
$$K_3 = f(x+\Delta x, u(x)-K_1\Delta x+2\,K_2\Delta x) \, .$$

In our case, $f(x, u) = -2u + x + 4$. At x = 0 (the initial state), and using $\Delta x = 0.2$, we have

$$K_1 = f(0, u(0)) = f(0, 1) = -2*1+0+4 = 2$$
$$K_2 = f(0.1, u(0)+2*0.2/2) = f(0.1, 1.2) = -2*1.2+0.1+4 = 1.7$$
$$K_3 = f(0.2, u(0)-2*0.2+2*1.7*0.2) = f(0.2, 1.28) = -2*1.28+0.2+4 = 1.64$$

Thus,

$$u(0.2) = u(0) + (1/6)*\,(2 + 4*1.7+ 1.64)*\,0.2 = \mathbf{1.348} \, .$$

(ii) 4th order Rugne-Kutta method

For a general ODE, $\frac{du}{dx} = f(x, u(x))$, the formula reads

$$u(x+\Delta x) = u(x) + (1/6)\,(K_1 + 2\,K_2 + 2\,K_3 + K_4)\,\Delta x\ ,$$
$$K_1 = f(x, u(x))\ ,$$
$$K_2 = f(x+\Delta x/2,\ u(x)+K_1\Delta x/2)\ ,$$
$$K_3 = f(x+\Delta x/2,\ u(x)+K_2\Delta x/2)\ ,$$
$$K_4 = f(x+\Delta x,\ u(x)+K_3\Delta x)$$

For our I.V.P., using $\Delta x = 0.2$, we have

$$K_1 = f(0, u(0)) = f(0, 1) = -2*1+0+4 = 2$$
$$K_2 = f(0.1, u(0)+2*0.2/2) = f(0.1, 1.2) = -2*1.2+0.1+4 = 1.7$$
$$K_3 = f(0.1, u(0)+1.7*0.2/2) = f(0.1, 1.17) = -2*1.17+0.1+4 = 1.76$$
$$K_4 = f(0.2, u(0)+1.76*0.2) = f(0.2, 1.352) = -2*1.352+0.2+4 = 1.496\ ,$$

which leads to

$$u(0.2) = u(0)+(1/6)*(2+2*1.7+2*1.76+1.496)*0.2 = \mathbf{1.3472}\ .$$

Note that the exact solution is $u(x) = -0.75\exp(-2x)+0.5x+1.75$, or $u(0.2) =$ **1.3472599...** The 4th order R-K method is more accurate than the 3rd order R-K method with the same Δx.