Fourier Series

Fourier series started life as a method to solve problems about the flow of heat through ordinary materials. It has grown so far that if you search our library's catalog for the keyword "Fourier" you will find 618 entries as of this date. It is a tool in abstract analysis and electromagnetism and statistics and radio communication and . . . . People have even tried to use it to analyze the stock market. (It didn't help.) The representation of musical sounds as sums of waves of various frequencies is an audible example. It provides an indispensible tool in solving partial differential equations, and a later chapter will show some of these tools at work.

5.1 Examples

The power series or Taylor series is based on the idea that you can write a general function as an infinite series of powers. The idea of Fourier series is that you can write a function as an infinite series of sines and cosines. You can also use functions other than trigonometric ones, but I'll leave that generalization aside for now, except to say that Legendre polynomials are an important example of functions used for such more general expansions.

An example: On the interval $0 < x < L$ the function x^2 varies from 0 to L^2. It can be written as the series of cosines

$$x^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_1^\infty \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{L}$$

$$= \frac{L^2}{3} - \frac{4L^2}{\pi^2} \left[\cos \frac{\pi x}{L} - \frac{1}{4} \cos \frac{2\pi x}{L} + \frac{1}{9} \cos \frac{3\pi x}{L} - \cdots \right] \tag{5.1}$$

To see if this is even plausible, examine successive partial sums of the series, taking one term, then two terms, etc. Sketch the graphs of these partial sums to see if they start to look like the function they are supposed to represent (left graph). The graphs of the series, using terms up to $n = 5$ do pretty well at representing the parabola.
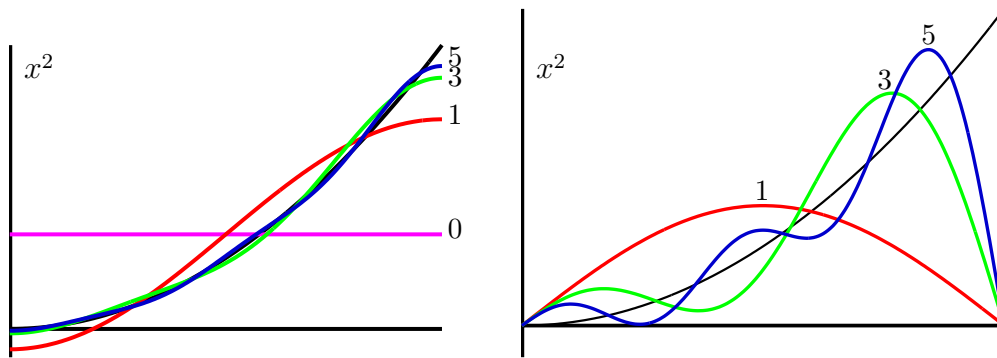
The same function can be written in terms of sines with another series:

$$x^2 = \frac{2L^2}{\pi} \sum_1^\infty \left[\frac{(-1)^{n+1}}{n} - \frac{2}{\pi^2 n^3} \left(1 - (-1)^n \right) \right] \sin \frac{n\pi x}{L} \tag{5.2}$$

and again you can see how the series behaves by taking one to several terms of the series. (right graph) The graphs show the parabola $y = x^2$ and partial sums of the two series with terms up to $n = 1$, 3, 5.

The second form doesn't work as well as the first one, and there's a reason for that. The sine functions all go to zero at $x = L$ and x^2 doesn't, making it hard for the sum of sines to approximate the desired function. They can do it, but it takes a lot more terms in the series to get a satisfactory result. The series Eq. (5.1) has terms that go to zero as $1/n^2$, while the terms in the series Eq. (5.2) go to zero only as $1/n$.*

5.2 Computing Fourier Series

How do you determine the details of these series starting from the original function? For the Taylor series, the trick was to assume a series to be an infinitely long polynomial and then to evaluate it (and its successive derivatives) at a point. You require that all of these values match those of the desired function at that one point. That method won't work in this case. (Actually I've read that it can work here too, but with a ridiculous amount of labor and some mathematically suspect procedures.)
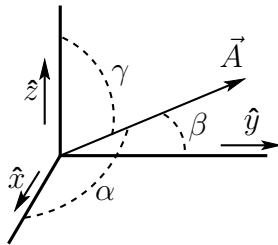
The idea of Fourier's procedure is like one that you can use to determine the components of a vector in three dimensions. You write such a vector as

$$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}$$

And then use the orthonormality of the basis vectors, $\hat{x} \cdot \hat{y} = 0$ etc. Take the scalar product of the preceding equation with $\hat{x}$.

$$\hat{x} \cdot \vec{A} = \hat{x} \cdot \left(A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z} \right) = A_x \qquad \text{and} \qquad \hat{y} \cdot \vec{A} = A_y \qquad \text{and} \qquad \hat{z} \cdot \vec{A} = A_z \qquad (5.3)$$

This lets you get all the components of $\vec{A}$. For example,

$$\hat{x} \cdot \vec{A} = A_x = A \cos\alpha$$
$$\hat{y} \cdot \vec{A} = A_y = A \cos\beta \qquad\qquad (5.4)$$
$$\hat{z} \cdot \vec{A} = A_z = A \cos\gamma$$

This shows the three direction cosines for the vector $\vec{A}$. You will occasionally see these numbers used to describe vectors in three dimensions, and it's easy to see that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

In order to stress the close analogy between this scalar product and what you do in Fourier series, I will introduce another notation for the scalar product. You don't typically see it in introductory courses for the simple reason that it isn't needed there. Here however it will turn out to be very useful, and in the next chapter you will see nothing *but* this notation. Instead of $\hat{x} \cdot \vec{A}$ or $\vec{A} \cdot \vec{B}$ you use $\langle \hat{x}, \vec{A} \rangle$ or $\langle \vec{A}, \vec{B} \rangle$. The angle bracket notation will make it very easy to generalize the idea of a dot product to cover other things. In this notation the above equations will appear as

$$\langle \hat{x}, \vec{A} \rangle = A \cos\alpha, \qquad \langle \hat{y}, \vec{A} \rangle = A \cos\beta, \qquad \langle \hat{z}, \vec{A} \rangle = A \cos\gamma$$

and they mean exactly the same thing as Eq. (5.4).

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* For animated sequences showing the convergence of some of these series, see
www.physics.miami.edu/nearing/mathmethods/animations.html

There are orthogonality relations similar to the ones for $\hat{x}$, $\hat{y}$, and $\hat{z}$, but for sines and cosines. Let n and m represent integers, then

$$\int_0^L dx\, \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases} \tag{5.5}$$

This is sort of like $\hat{x} \cdot \hat{z} = 0$ and $\hat{y} \cdot \hat{y} = 1$, where the analog of $\hat{x}$ is $\sin \pi x/L$ and the analog of $\hat{y}$ is $\sin 2\pi x/L$. The biggest difference is that it doesn't stop with three vectors in the basis; it keeps on with an infinite number of values of n and the corresponding different sines. There are an infinite number of very different possible functions, so you need an infinite number of basis functions in order to express a general function as a sum of them. The integral Eq. (5.5) is a continuous analog of the coordinate representation of the common dot product. The sum over three terms $A_x B_x + A_y B_y + A_z B_z$ becomes a sum (integral) over a continuous index, the integration variable. By using this integral as a generalization of the ordinary scalar product, you can say that $\sin(\pi x/L)$ and $\sin(2\pi x/L)$ are *orthogonal*. Let i be an index taking on the values x, y, and z, then the notation A_i is a function of the variable i. In this case the independent variable takes on just three possible values instead of the infinite number in Eq. (5.5).

How do you derive an identity such as Eq. (5.5)? The first method is just straight integration, using the right trigonometric identities. The easier (and more general) method can wait for a few pages.

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y, \qquad \text{subtract:} \qquad \cos(x - y) - \cos(x + y) = 2 \sin x \sin y$$

Use this in the integral.

$$2 \int_0^L dx\, \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \int_0^L dx\, \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right]$$

Now do the integral, assuming $n \neq m$ and that n and m are positive integers.

$$= \frac{L}{(n-m)\pi} \sin\left(\frac{(n-m)\pi x}{L}\right) - \frac{L}{(n+m)\pi} \sin\left(\frac{(n+m)\pi x}{L}\right) \Bigg|_0^L = 0 \tag{5.6}$$

Why assume that the integers are positive? Aren't the negative integers allowed too? Yes, but they aren't needed. Put $n = -1$ into $\sin(n\pi x/L)$ and you get the same function as for $n = +1$, but turned upside down. It isn't an independent function, just -1 times what you already have. Using it would be sort of like using for your basis not only $\hat{x}$, $\hat{y}$, and $\hat{z}$ but $-\hat{x}$, $-\hat{y}$, and $-\hat{z}$ too. Do the $n = m$ case of the integral yourself.

Computing an Example

For a simple example, take the function $f(x) = 1$, the constant on the interval $0 < x < L$, and assume that there is a series representation for f on this interval.

$$1 = \sum_1^\infty a_n \sin\left(\frac{n\pi x}{L}\right) \qquad (0 < x < L) \tag{5.7}$$

Multiply both sides by the sine of $m\pi x/L$ and integrate from 0 to L.

$$\int_0^L dx\, \sin\left(\frac{m\pi x}{L}\right) 1 = \int_0^L dx\, \sin\left(\frac{m\pi x}{L}\right) \sum_{n=1}^\infty a_n \sin\left(\frac{n\pi x}{L}\right) \tag{5.8}$$

Interchange the order of the sum and the integral, and the integral that shows up is the orthogonality integral derived just above. When you use the orthogonality of the sines, only one term in the infinite series survives.

$$\int_0^L dx \, \sin\left(\frac{m\pi x}{L}\right) 1 = \sum_{n=1}^{\infty} a_n \int_0^L dx \, \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} a_n \cdot \begin{cases} 0 & (n \neq m) \\ L/2 & (n = m) \end{cases} \tag{5.9}$$

$$= a_m L/2.$$

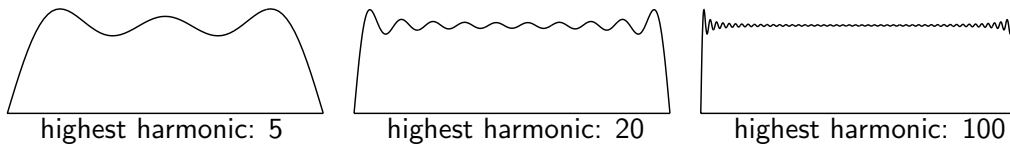Now all you have to do is to evaluate the integral on the left.

$$\int_0^L dx \, \sin\left(\frac{m\pi x}{L}\right) 1 = \frac{L}{m\pi}\left[-\cos\frac{m\pi x}{L}\right]_0^L = \frac{L}{m\pi}\left[1 - (-1)^m\right]$$

This is zero for even m, and when you equate it to (5.9) you get

$$a_m = \frac{4}{m\pi} \qquad \text{for } m \text{ odd}$$

You can relabel the indices so that the sum shows only odd integers $m = 2k+1$ and the Fourier series is

$$\frac{4}{\pi} \sum_{m \text{ odd} > 0} \frac{1}{m} \sin\frac{m\pi x}{L} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\frac{(2k+1)\pi x}{L} = 1, \qquad (0 < x < L) \tag{5.10}$$

highest harmonic: 5   highest harmonic: 20   highest harmonic: 100

The graphs show the sum of the series up to $2k + 1 = 5$, 19, 99 respectively. It is not a very rapidly converging series, but it's a start. You can see from the graphs that near the end of the interval, where the function is discontinuous, the series has a hard time handling the jump. The resulting overshoot is called the Gibbs phenomenon, and it is analyzed in section 5.7.

Notation
The point of introducing that other notation for the scalar product comes right here. The same notation is used for these integrals. In this context define

$$\langle f, g \rangle = \int_0^L dx \, f(x)^* g(x) \tag{5.11}$$

and it will behave just the same way that $\vec{A} \cdot \vec{B}$ does. Eq. (5.5) then becomes

$$\langle u_n, u_m \rangle = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases} \qquad \text{where} \qquad u_n(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{5.12}$$

$$\text{precisely analogous to} \qquad \langle \hat{x}, \hat{x} \rangle = 1 \qquad \text{and} \qquad \langle \hat{y}, \hat{z} \rangle = 0$$

These u_n are orthogonal to each other even though they aren't normalized to one the way that $\hat{x}$ and $\hat{y}$ are, but that turns out not to matter. $\langle u_n, u_n \rangle = L/2$ instead of $= 1$, so you simply keep track of it.

(What happens to the series Eq. (5.7) if you multiply every u_n by 2? Nothing, because the coefficients a_n get multiplied by $1/2$.)

The Fourier series manipulations, Eqs. (5.7), (5.8), (5.9), become

$$1 = \sum_1^\infty a_n u_n \qquad \text{then} \qquad \langle u_m, 1 \rangle = \Big\langle u_m, \sum_1^\infty a_n u_n \Big\rangle = \sum_{n=1}^\infty a_n \langle u_m, u_n \rangle = a_m \langle u_m, u_m \rangle \quad (5.13)$$

This is far more compact than you see in the steps between Eq. (5.7) and Eq. (5.10). You *still* have to evaluate the integrals $\langle u_m, 1 \rangle$ and $\langle u_m, u_m \rangle$, but when you master this notation you'll likely make fewer mistakes in figuring out what integral you have to do. Again, you can think of Eq. (5.11) as a continuous analog of the discrete sum of three terms, $\langle \vec{A}, \vec{B} \rangle = A_x B_x + A_y B_y + A_z B_z$.

The analogy between the vectors such as $\hat{x}$ and functions such as sine is really far deeper, and it is central to the subject of the next chapter. In order not to get confused by the notation, you have to distinguish between a whole function f, and the value of that function at a point, $f(x)$. The former is the whole graph of the function, and the latter is one point of the graph, analogous to saying that $\vec{A}$ is the whole vector and A_y is one of its components.

The scalar product notation defined in Eq. (5.11) is not necessarily restricted to the interval $0 < x < L$. Depending on context it can be over any interval that you happen to be considering at the time. In Eq. (5.11) there is a complex conjugation symbol. The functions here have been real, so this made no difference, but you will often deal with complex functions and then the fact that the notation $\langle f, g \rangle$ includes a conjugation is important. This notation is a special case of the general development that will come in section 6.6. The basis vectors such as $\hat{x}$ are conventionally normalized to one, $\hat{x} \cdot \hat{x} = 1$, but you don't have to require it even there, and in the context of Fourier series it would clutter up the notation to require $\langle u_n, u_n \rangle = 1$, so I don't bother.

Some Examples
To get used to this notation, try showing that these pairs of functions are orthogonal on the interval $0 < x < L$. Sketch graphs of both functions in every case.

$$\langle x, L - \tfrac{3}{2}x \rangle = 0 \qquad \langle \sin \pi x / L, \cos \pi x / L \rangle = 0 \qquad \langle \sin 3\pi x / L, L - 2x \rangle = 0$$

The notation has a complex conjugation built into it, but these examples are all real. What if they aren't? Show that these are orthogonal too. How do you graph these? Not easily.*

$$\langle e^{2i\pi x/L}, e^{-2i\pi x/L} \rangle = 0 \qquad \langle L - \tfrac{1}{4}(7+i)x, L + \tfrac{3}{2}ix \rangle = 0$$

Extending the function
In Equations (5.1) and (5.2) the original function was specified on the interval $0 < x < L$. The two Fourier series that represent it can be evaluated for *any* x. Do they equal x^2 everywhere? No. The first series involves only cosines, so it is an even function of x, but it's periodic: $f(x + 2L) = f(x)$. The second series has only sines, so it's odd, and it too is periodic with period $2L$.

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* but see if you can find a copy of the book by Jahnke and Emde, published long before computers. They show examples. Also check out
www.geom.uiuc.edu/~banchoff/script/CFGInd.html or
www.math.ksu.edu/~bennett/jomacg/
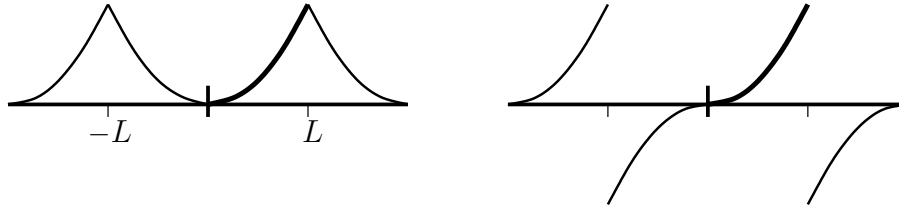
Here the discontinuity in the sine series is more obvious, a fact related to its slower convergence.

5.3 Choice of Basis

When you work with components of vectors in two or three dimensions, you will choose the basis that is most convenient for the problem you're working with. If you do a simple mechanics problem with a mass moving on an incline, you can choose a basis $\hat{x}$ and $\hat{y}$ that are arranged horizontally and vertically. OR, you can place them at an angle so that they point down the incline and perpendicular to it. The latter is often a simpler choice in that type of problem.

The same applies to Fourier series. The interval on which you're working is not necessarily from zero to L, and even on the interval $0 < x < L$ you can choose many sets of function for a basis:

$\sin n\pi x/L \quad (n = 1, 2, \ldots)$ as in equations (5.10) and (5.2), or you can choose a basis

$\cos n\pi x/L \quad (n = 0, 1, 2, \ldots)$ as in Eq. (5.1), or you can choose a basis

$\sin(n + \tfrac{1}{2})\pi x/L \quad (n = 0, 1, 2, \ldots)$, or you can choose a basis

$e^{2\pi i n x/L} \quad (n = 0, \pm 1, \pm 2, \ldots)$, or an infinite number of other possibilities.

In order to use any of these you need a relation such as Eq. (5.5) for each separate case. That's a lot of integration. You need to do it for any interval that you may need and that's even more integration. Fortunately there's a way out:

Fundamental Theorem

If you want to show that each of these respective choices provides an orthogonal set of functions you can integrate every special case as in Eq. (5.6), *or* you can do all the cases at once by deriving an important theorem. This theorem starts from the fact that all of these sines and cosines and complex exponentials satisfy the same differential equation, $u'' = \lambda u$, where λ is some constant, different in each case. If you studied section 4.5, you saw how to derive properties of trigonometric functions simply by examining the differential equation that they satisfy. If you didn't, now might be a good time to look at it, because this is more of the same. (I'll wait.)

You have two functions u_1 and u_2 that satisfy

$$u_1'' = \lambda_1 u_1 \qquad \text{and} \qquad u_2'' = \lambda_2 u_2$$

Make no assumption about whether the λ's are positive or negative or even real. The u's can also be complex. Multiply the first equation by u_2^* and the second by u_1^*, then take the complex conjugate of the second product.

$$u_2^* u_1'' = \lambda_1 u_2^* u_1 \qquad \text{and} \qquad u_1 u_2^{*''} = \lambda_2^* u_1 u_2^*$$

Subtract the equations.

$$u_2^* u_1'' - u_1 u_2^{*''} = (\lambda_1 - \lambda_2^*) u_2^* u_1$$

Integrate from a to b

$$\int_a^b dx \left(u_2^* u_1'' - u_1 u_2^{*''} \right) = (\lambda_1 - \lambda_2^*) \int_a^b dx\, u_2^* u_1 \tag{5.14}$$

Now do two partial integrations. Work on the second term on the left:

$$\int_a^b dx\, u_1 u_2^{*\prime\prime} = u_1 u_2^{*\prime}\Big|_a^b - \int_a^b dx\, u_1' u_2^{*\prime} = u_1 u_2^{*\prime}\Big|_a^b - u_1' u_2^*\Big|_a^b + \int_a^b dx\, u_1'' u_2^*$$

Put this back into the Eq. (5.14) and the integral terms cancel, leaving

$$\boxed{u_1' u_2^* - u_1 u_2^{*\prime}\Big|_a^b = (\lambda_1 - \lambda_2^*)\int_a^b dx\, u_2^* u_1} \tag{5.15}$$

This is the central identity from which all the orthogonality relations in Fourier series derive. It's even more important than that because it tells you what types of boundary conditions you can use in order to get the desired orthogonality relations. (It tells you even more than that, as it tells you how to compute the adjoint of the second derivative operator. But not now — save that for later.) The expression on the left side of the equation has a name: "bilinear concomitant."

You can see how this is related to the work with the functions $\sin(n\pi x/L)$. They satisfy the differential equation $u'' = \lambda u$ with $\lambda = -n^2\pi^2/L^2$. The interval in that case was $0 < x < L$ for $a < x < b$.

There are generalizations of this theorem that you will see in places such as problems 6.16 and 6.17 and 10.21. In those extensions these same ideas will allow you to handle Legendre polynomials and Bessel functions and Ultraspherical polynomials and many other functions in just the same way that you handle sines and cosines. That development comes under the general name Sturm-Liouville theory.

The key to using this identity will be to figure out what sort of boundary conditions will cause the left-hand side to be zero. For example if $u(a) = 0$ and $u(b) = 0$ then the left side vanishes. These are not the only possible boundary conditions that make this work; there are several other common cases soon to appear.

The first consequence of Eq. (5.15) comes by taking a special case, the one in which the two functions u_1 and u_2 are in fact the same function. If the boundary conditions make the left side zero then

$$0 = (\lambda_1 - \lambda_1^*)\int_a^b dx\, u_1^*(x) u_1(x)$$

The λ's are necessarily the same because the u's are. The only way the product of two numbers can be zero is if one of them is zero. The integrand, $u_1^*(x)u_1(x)$ is always non-negative and is continuous, so the integral can't be zero unless the function u_1 is identically zero. As that would be a trivial case, assume it's not so. This then implies that the other factor, $(\lambda_1 - \lambda_1^*)$ must be zero, and this says that the constant λ_1 is real. Yes, $-n^2\pi^2/L^2$ is real.

[To use another language that will become more familiar later, λ is an eigenvalue and d^2/dx^2 with these boundary conditions is an operator. This calculation guarantees that the eigenvalue is real.]

Now go back to the more general case of two different functions, and drop the complex conjugation on the λ's.

$$0 = (\lambda_1 - \lambda_2)\int_a^b dx\, u_2^*(x) u_1(x)$$

This says that if the boundary conditions on u make the left side zero, then for two solutions with different eigenvalues (λ's) the orthogonality integral is zero. Eq. (5.5) is a special case of the following equation.

$$\text{If}\quad \lambda_1 \neq \lambda_2, \qquad \text{then}\qquad \langle u_2, u_1\rangle = \int_a^b dx\, u_2^*(x) u_1(x) = 0 \tag{5.16}$$

Apply the Theorem

As an example, carry out a full analysis of the case for which $a = 0$ and $b = L$, and for the boundary conditions $u(0) = 0$ and $u(L) = 0$. The parameter λ is positive, zero, or negative. If $\lambda > 0$, then set $\lambda = k^2$ and

$$u(x) = A \sinh kx + B \cosh kx, \qquad \text{then} \qquad u(0) = B = 0$$
$$\text{and so} \qquad u(L) = A \sinh kL = 0 \Rightarrow A = 0$$

No solutions there, so try $\lambda = 0$

$$u(x) = A + Bx, \qquad \text{then} \qquad u(0) = A = 0 \qquad \text{and so} \qquad u(L) = BL = 0 \Rightarrow B = 0$$

No solutions here either. Try $\lambda < 0$, setting $\lambda = -k^2$.

$$u(x) = A \sin kx + B \cos kx, \qquad \text{then} \qquad u(0) = 0 = B, \qquad \text{so} \qquad u(L) = A \sin kL = 0$$

Now there are many solutions because $\sin n\pi = 0$ allows $k = n\pi/L$ with n any integer. But, $\sin(-x) = -\sin(x)$ so negative integers just reproduce the same functions as do the positive integers; they are redundant and you can eliminate them. The complete set of solutions to the equation $u'' = \lambda u$ with these boundary conditions has $\lambda_n = -n^2\pi^2/L^2$ and reproduces the result of the explicit integration as in Eq. (5.6).

$$u_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1,\ 2,\ 3, \ldots \qquad \text{and}$$

$$\langle u_n, u_m \rangle = \int_0^L dx \, \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = 0 \qquad \text{if} \qquad n \neq m \qquad (5.17)$$

There are other choices of boundary condition that will make the bilinear concomitant vanish. (Verify these!) For example

$$u(0) = 0, \qquad u'(L) = 0 \qquad \text{gives} \qquad u_n(x) = \sin\left(n + \tfrac{1}{2}\right)\pi x/L \qquad n = 0,\ 1,\ 2,\ 3, \ldots$$

and without further integration you have the orthogonality integral for non-negative integers n and m

$$\langle u_n, u_m \rangle = \int_0^L dx \, \sin\left(\frac{(n + \tfrac{1}{2})\pi x}{L}\right) \sin\left(\frac{(m + \tfrac{1}{2})\pi x}{L}\right) = 0 \qquad \text{if} \qquad n \neq m \qquad (5.18)$$

A very common choice of boundary conditions is

$$u(a) = u(b), \qquad u'(a) = u'(b) \qquad \text{(periodic boundary conditions)} \qquad (5.19)$$

It is often more convenient to use complex exponentials in this case (though of course not necessary). On $0 < x < L$

$$u(x) = e^{ikx}, \qquad \text{where} \qquad k^2 = -\lambda \qquad \text{and} \qquad u(0) = 1 = u(L) = e^{ikL}$$

The periodic behavior of the exponential implies that $kL = 2n\pi$. The condition that the derivatives match at the boundaries makes no further constraint, so the basis functions are

$$u_n(x) = e^{2\pi i n x/L}, \qquad (n = 0,\ \pm 1, \pm 2,\ \ldots) \qquad (5.20)$$

Notice that in this case the index n runs over all positive and negative numbers and zero, not just the positive integers. The functions $e^{2\pi inx/L}$ and $e^{-2\pi inx/L}$ are independent, unlike the case of the sines discussed above. Without including both of them you don't have a basis and can't do Fourier series. If the interval is symmetric about the origin as it often is, $-L < x < +L$, the conditions are

$$u(-L) = e^{-ikL} = u(+L) = e^{+ikL}, \qquad \text{or} \qquad e^{2ikL} = 1 \tag{5.21}$$

This says that $2kL = 2n\pi$, so

$$u_n(x) = e^{n\pi ix/L}, \qquad (n = 0, \pm 1, \pm 2, \ldots) \qquad \text{and} \qquad f(x) = \sum_{-\infty}^{\infty} c_n u_n(x)$$

The orthogonality properties determine the coefficients:

$$\langle u_m, f \rangle = \Big\langle u_m, \sum_{-\infty}^{\infty} c_n u_n \Big\rangle = c_m \langle u_m, u_m \rangle$$

$$\int_{-L}^{L} dx \, e^{-m\pi ix/L} f(x) = c_m \langle u_m, u_m \rangle$$

$$= c_m \int_{-L}^{L} dx \, e^{-m\pi ix/L} e^{+m\pi ix/L} = c_m \int_{-L}^{L} dx \, 1 = 2L c_m$$

In this case, sometimes the real form of this basis is more convenient and you can use the combination of the two sets u_n and v_n, where

$$u_n(x) = \cos(n\pi x/L), \ (n = 0, 1, 2, \ldots)$$
$$v_n(x) = \sin(n\pi x/L), \ (n = 1, 2, \ldots) \tag{5.22}$$

$$\langle u_n, u_m \rangle = 0 \quad (n \neq m), \qquad \langle v_n, v_m \rangle = 0 \quad (n \neq m), \qquad \langle u_n, v_m \rangle = 0 \quad (\text{all } n, m)$$

and the Fourier series is a sum such as $f(x) = \sum_0^\infty a_n u_n + \sum_1^\infty b_n v_n$.

There are an infinite number of other choices, a few of which are even useful, *e.g.*

$$u'(a) = 0 = u'(b) \tag{5.23}$$

Take the same function as in Eq. (5.7) and try a different basis. Choose the basis for which the boundary conditions are $u(0) = 0$ and $u'(L) = 0$. This gives the orthogonality conditions of Eq. (5.18). The general structure is always the same.

$$f(x) = \sum a_n u_n(x), \qquad \text{and use} \qquad \langle u_m, u_n \rangle = 0 \ (n \neq m)$$

Take the scalar product of this equation with u_m to get

$$\langle u_m, f \rangle = \Big\langle u_m, \sum a_n u_n \Big\rangle = a_m \langle u_m, u_m \rangle \tag{5.24}$$

This is exactly as before in Eq. (5.13), but with a different basis. To evaluate it you still have to do the integrals.

$$\langle u_m, f \rangle = \int_0^L dx \, \sin\left(\frac{(m + 1/2)\pi x}{L}\right) 1 = a_m \int_0^L dx \, \sin^2\left(\frac{(m + 1/2)\pi x}{L}\right) = a_m \langle u_m, u_m \rangle$$

$$\frac{L}{(m + 1/2)\pi} \Big[1 - \cos\big((m + 1/2)\pi\big)\Big] = \frac{L}{2} a_m$$

$$a_m = \frac{4}{(2m + 1)\pi}$$

Then the series is

$$\frac{4}{\pi} \left[\sin\frac{\pi x}{2L} + \frac{1}{3}\sin\frac{3\pi x}{2L} + \frac{1}{5}\sin\frac{5\pi x}{2L} + \cdots\right] \tag{5.25}$$

5.4 Musical Notes

Different musical instruments sound different even when playing the same note. You won't confuse the sound of a piano with the sound of a guitar, and the reason is tied to Fourier series. The note middle C has a frequency that is 261.6 Hz on the standard equal tempered scale. The angular frequency is then 2π times this, or 1643.8 radians/sec. Call it $\omega_0 = 1644$. When you play this note on any musical instrument, the result is always a combination of many frequencies, this one and many multiples of it. A pure frequency has just ω_0, but a real musical sound has many harmonics: ω_0, $2\omega_0$, $3\omega_0$, *etc.*

$$\text{Instead of} \quad e^{i\omega_0 t} \qquad \text{an instrument produces} \qquad \sum_{n=1}^{?} a_n e^{ni\omega_0 t} \tag{5.26}$$

A pure frequency is the sort of sound that you hear from an electronic audio oscillator, and it's not very interesting. Any real musical instrument will have at least a few and usually many frequencies combined to make what you hear.

Why write this as a complex exponential? A sound wave is a real function of position and time, the pressure wave, but it's easier to manipulate complex exponentials than sines and cosines, so when I write this, I really mean to take the real part for the physical variable, the pressure variation. The imaginary part is carried along to be discarded later. Once you're used to this convention you don't bother writing the "real part understood" anywhere — it's understood.

$$p(t) = \Re \sum_{n=1}^{?} a_n e^{ni\omega_0 t} = \sum_{n=1}^{?} |a_n| \cos\left(n\omega_0 t + \phi_n\right) \qquad \text{where} \qquad a_n = |a_n| e^{i\phi_n} \tag{5.27}$$

I wrote this using the periodic boundary conditions of Eq. (5.19). The period is the period of the lowest frequency, $T = 2\pi/\omega_0$.

A flute produces a combination of frequencies that is mostly concentrated in a small number of harmonics, while a violin or reed instrument produces a far more complex combination of frequencies. The size of the coefficients a_n in Eq. (5.26) determines the quality of the note that you hear, though oddly enough its *phase*, ϕ_n, doesn't have an effect on your perception of the sound.

These represent a couple of cycles of the sound of a clarinet. The left graph is about what the wave output of the instrument looks like, and the right graph is what the graph would look like if I add a random phase, ϕ_n, to each of the Fourier components of the sound as in Eq. (5.27). They may look very different, but to the human ear they sound alike.

You can hear examples of the sound of Fourier series online via the web site:
courses.ee.sun.ac.za/Stelsels_en_Seine_315/wordpress/wp-content/uploads/jhu-signals/
and Listen to Fourier Series

You can hear the (lack of) effect of phase on sound. You can also synthesize your own series and hear what they sound like under such links as "Fourier synthese" and "Harmonics applet" found on this page. You can back up from this link to larger topics by using the links shown in the left column of the web page.

Real musical sound is of course more than just these Fourier series. At the least, the Fourier coefficients, a_n, are themselves functions of time. The time scale on which they vary is however much longer than the basic period of oscillation of the wave. That means that it makes sense to treat them as (almost) constant when you are trying to describe the harmonic structure of the sound. Even the lowest pitch notes you can hear are at least 20 Hz, and few home sound systems can produce frequencies nearly that low. Musical notes change on time scales much greater than $1/20$ or $1/100$ of a second, and this allows you to treat the notes by Fourier series even though the Fourier coefficients are themselves time-dependent. The attack and the decay of the note greatly affects our perception of it, and that is described by the time-varying nature of these coefficients.*

Parseval's Identity
Let u_n be the set of orthogonal functions that follow from your choice of boundary conditions.

$$f(x) = \sum_n a_n u_n(x)$$

Evaluate the integral of the absolute square of f over the domain.

$$\int_a^b dx \, |f(x)|^2 = \int_a^b dx \left[\sum_m a_m u_m(x) \right]^* \left[\sum_n a_n u_n(x) \right]$$

$$= \sum_m a_m^* \sum_n a_n \int_a^b dx \, u_m(x)^* u_n(x) = \sum_n |a_n|^2 \int_a^b dx \, |u_n(x)|^2$$

In the more compact notation this is

$$\langle f, f \rangle = \left\langle \sum_m a_m u_m, \sum_n a_n u_n \right\rangle = \sum_{m,n} a_m^* a_n \langle u_m, u_n \rangle = \sum_n |a_n|^2 \langle u_n, u_n \rangle \qquad (5.28)$$

The first equation is nothing more than substituting the series for f. The second moved the integral under the summation. The third equation uses the fact that all these integrals are zero except for the ones with $m = n$. That reduces the double sum to a single sum. If you have chosen to normalize all of the functions u_n so that the integrals of $|u_n(x)|^2$ are one, then this relation takes on a simpler appearance. This is sometimes convenient.

What does this say if you apply it to a series I've just computed? Take Eq. (5.10) and see what it implies.

$$\langle f, f \rangle = \int_0^L dx \, 1 = L = \sum_{k=0}^{\infty} |a_k|^2 \langle u_n, u_n \rangle =$$

$$\sum_{k=0}^{\infty} \left(\frac{4}{\pi(2k+1)} \right)^2 \int_0^L dx \, \sin^2 \left(\frac{(2k+1)\pi x}{L} \right) = \sum_{k=0}^{\infty} \left(\frac{4}{\pi(2k+1)} \right)^2 \frac{L}{2}$$

Rearrange this to get

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \qquad (5.29)$$

---

* For an enlightening web page, including a complete and impressively thorough text on mathematics and music, look up the book by David Benson. It is available both in print from Cambridge Press and online. www.abdn.ac.uk/~mth192/ (University of Aberdeen)

A bonus. You have the sum of this infinite series, a result that would be quite perplexing if you see it without knowing where it comes from. While you have it in front of you, what do you get if you simply *evaluate* the infinite series of Eq. (5.10) at $L/2$. The answer is 1, but what is the other side?

$$1 = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin \frac{(2k+1)\pi(L/2)}{L} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1}(-1)^k$$

$$\text{or} \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

But does it Work?

If you are in the properly skeptical frame of mind, you may have noticed a serious omission on my part. I've done all this work showing how to get orthogonal functions and to manipulate them to derive Fourier series for a general function, but when did I show that this actually works? Never. How do I know that a general function, even a well-behaved general function, can be written as such a series? I've proved that the set of functions $\sin(n\pi x/L)$ are orthogonal on $0 < x < L$, but that's not good enough.

Maybe a clever mathematician will invent a new function that I haven't thought of and that will be orthogonal to all of these sines and cosines that I'm trying to use for a basis, just as $\hat{k}$ is orthogonal to $\hat{i}$ and $\hat{j}$. It won't happen. There are proper theorems that specify the conditions under which all of this Fourier manipulation works. Dirichlet worked out the key results, which are found in many advanced calculus texts.

For example if the function is continuous with a continuous derivative on the interval $0 \le x \le L$ then the Fourier series will exist, will converge, and will converge to the specified function (except maybe at the endpoints). If you allow it to have a finite number of finite discontinuities but with a continuous derivative in between, then the Fourier series will converge and will (except maybe at the discontinuities) converge to the specified function. At these discontinuities it will converge to the average value taken from the left and from the right. There are a variety of other sufficient conditions that you can use to insure that all of this stuff works, but I'll leave that to the advanced calculus books.

5.5 Periodically Forced ODE's

If you have a harmonic oscillator with an added external force, such as Eq. (4.12), there are systematic ways to solve it, such as those found in section 4.2. One part of the problem is to find a solution to the inhomogeneous equation, and if the external force is simple enough you can do this easily. Suppose though that the external force is complicated *but periodic*, as for example when you're pushing a child on a swing.

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_{\text{ext}}(t)$$

That the force is periodic means $F_{\text{ext}}(t) = F_{\text{ext}}(t+T)$ for all times t. The period is T.

Pure Frequency Forcing

Before attacking the general problem, look at a simple special case. Take the external forcing function to be $F_0 \cos \omega_e t$ where this frequency is $\omega_e = 2\pi/T$. This equation is now

$$m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = F_0 \cos \omega_e t = \frac{F_0}{2}\left[e^{i\omega_e t} + e^{-i\omega_e t}\right] \tag{5.30}$$

Find a solution corresponding to each term separately and add the results. To get an exponential out, put an exponential in.

$$\text{for} \quad m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = e^{i\omega_e t} \quad \text{assume} \quad x_{\text{inh}}(t) = Ae^{i\omega_e t}$$

Substitute the assumed form and it will determine A.

$$\left[m(-\omega_e^2) + b(i\omega_e) + k\right]Ae^{i\omega_e t} = e^{i\omega_e t}$$

This tells you the value of A is

$$A = \frac{1}{-m\omega_e^2 + b\,i\omega_e + k} \tag{5.31}$$

The other term in Eq. (5.30) simply changes the sign in front of i everywhere. The total solution for Eq. (5.30) is then

$$x_{\text{inh}}(t) = \frac{F_0}{2}\left[\frac{1}{-m\omega_e^2 + b\,i\omega_e + k}e^{i\omega_e t} + \frac{1}{-m\omega_e^2 - b\,i\omega_e + k}e^{-i\omega_e t}\right] \tag{5.32}$$

This is the sum of a number and its complex conjugate, so it's real. You can rearrange it so that it looks a lot simpler, but there's no need to do that right now. Instead I'll look at what it implies for certain values of the parameters.

Suppose that the viscous friction is small (b is small). If the forcing frequency, ω_e is such that $-m\omega_e^2 + k = 0$, or is even close to zero, the denominators of the two terms become very small. This in turn implies that the response of x to the oscillating force is huge. *Resonance.* See problem 5.27. In a contrasting case, look at ω_e very large. Now the response of the mass is very small; it barely moves.
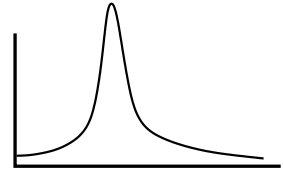
General Periodic Force
Now I'll go back to the more general case of a periodic forcing function, but not one that is simply a cosine. If a function is periodic I can use Fourier series to represent it on the whole axis. The basis to use will of course be the one with periodic boundary conditions (what else?). Use complex exponentials, then

$$u(t) = e^{i\omega t} \qquad \text{where} \qquad e^{i\omega(t+T)} = e^{i\omega t}$$

This is just like Eq. (5.20) but with t instead of x, so

$$u_n(t) = e^{2\pi int/T}, \qquad (n = 0, \pm 1, \ldots) \tag{5.33}$$

Let $\omega_e = 2\pi/T$, and this is

$$u_n(t) = e^{in\omega_e t}$$

The external force can now be represented by the Fourier series

$$F_{\text{ext}}(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{ik\omega_e t}, \qquad \text{where}$$

$$\left\langle e^{in\omega_e t}, \sum_{k=-\infty}^{\infty} a_k\, e^{ik\omega_e t}\right\rangle = a_n\, T = \left\langle e^{in\omega_e t}, F_{\text{ext}}(t)\right\rangle = \int_0^T dt\, e^{-in\omega_e t}\, F_{\text{ext}}(t)$$

(Don't forget the implied complex conjugation in the definition of the scalar product, $\langle\,,\,\rangle$, Eq. (5.11)) Because the force is periodic, any other time interval of duration T is just as good, perhaps $-T/2$ to $+T/2$ if that's more convenient.

How does this solve the differential equation? Plug in.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = \sum_{n=-\infty}^{\infty} a_n\, e^{in\omega_e t} \tag{5.34}$$

All there is to do now is to solve for an inhomogeneous solution one term at a time and then to add the results. Take one term alone on the right:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = e^{in\omega_e t}$$

This is what I just finished solving a few lines ago, Eq. (5.31), but with $n\omega_e$ instead of simply ω_e. The inhomogeneous solution is the sum of the solutions from each term.

$$x_{inh}(t) = \sum_{n=-\infty}^{\infty} a_n \frac{1}{-m(n\omega_e)^2 + bin\omega_e + k} e^{ni\omega_e t} \tag{5.35}$$

Suppose for example that the forcing function is a simple square wave.

$$F_{ext}(t) = \begin{cases} F_0 & (0 < t < T/2) \\ -F_0 & (T/2 < t < T) \end{cases} \quad \text{and} \quad F_{ext}(t+T) = F_{ext}(t) \tag{5.36}$$

The Fourier series for this function is one that you can do in problem 5.12. The result is

$$F_{ext}(t) = F_0\frac{2}{\pi i} \sum_{n \text{ odd}} \frac{1}{n} e^{ni\omega_e t} \tag{5.37}$$

The solution corresponding to Eq. (5.35) is now

$$x_{inh}(t) = F_0\frac{1}{2\pi i} \sum_{n \text{ odd}} \frac{1}{\left(-m(n\omega_e)^2 + ibn\omega_e + k\right)} \frac{1}{n} e^{ni\omega_e t} \tag{5.38}$$

A real force ought to give a real result; does this? Yes. For every positive n in the sum, there is a corresponding negative one and the sum of those two is real. You can see this because every n that appears is either squared or is multiplied by an "i." When you add the $n = +5$ term to the $n = -5$ term it's adding a number to its own complex conjugate, and that's real.

What peculiar features does this result imply? With the simply cosine force the phenomenon of resonance occurred, in which the response to a small force at a frequency that matched the intrinsic frequency $\sqrt{k/m}$ produced a disproportionately large response. What other things happen here?

The natural frequency of the system is (for small damping) still $\sqrt{k/m}$. Look to see where a denominator in Eq. (5.38) can become very small. This time it is when $-m(n\omega_e)^2 + k = 0$. This is not only when the external frequency ω_e matches the natural frequency; it's when $n\omega_e$ matches it. If the natural frequency is $\sqrt{k/m} = 100$ radians/sec you get a big response if the forcing frequency is 100 radians/sec or 33 radians/sec or 20 radians/sec or 14 radians/sec etc. What does this mean? The square wave in Eq. (5.36) contains many frequencies. It contains more than just the main frequency $2\pi/T$, it contains 3 times this and 5 times it and many higher frequencies. When any one of these harmonics matches the natural frequency you will have the large resonant response.

Not only do you get a large response, look at the way the mass oscillates. If the force has a square wave frequency 20 radians/sec, the mass responds* with a large sinusoidal oscillation at a frequency 5 times higher — 100 radians/sec.

---

* The next time you have access to a piano, gently depress a key without making a sound, then strike the key one octave lower. Release the lower key and listen to the sound of the upper note. Then try it with an interval of an octave plus a fifth.

5.6 Return to Parseval

When you have a periodic wave such as a musical note, you can Fourier analyze it. The boundary conditions to use are naturally the periodic ones, Eq. (5.20) or (5.33), so that

$$f(t) = \sum_{-\infty}^{\infty} a_n e^{in\omega_0 t}$$

If this represents the sound of a flute, the amplitudes of the higher frequency components (the a_n) drop off rapidly with n. If you are hearing an oboe or a violin the strength of the higher components is greater.

  If this function represents the sound wave as received by your ear, the power that you receive is proportional to the square of f. If f represent specifically the pressure disturbance in the air, the intensity (power per area) carried by the wave is $f(t)^2 v/B$ where v is the speed of the wave and B is the bulk modulus of the air. The key property of this is that it is proportional to the square of the wave's amplitude. That's the same relation that occurs for light or any other wave. Up to a known factor then, the power received by the ear is proportional to $f(t)^2$.

  This time average of the power is (up to that constant factor that I'm ignoring)

$$\langle f^2 \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt\, f(t)^2$$

Now put the Fourier series representation of the sound into the integral to get

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \left[\sum_{-\infty}^{\infty} a_n e^{in\omega_0 t} \right]^2$$

The sound $f(t)$ is real, so by problem 5.11, $a_{-n} = a_n^*$. Also, using the result of problem 5.18 the time average of $e^{i\omega t}$ is zero unless $\omega = 0$; then it's one.

$$\begin{aligned}
\langle f^2 \rangle &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \left[\sum_n a_n e^{in\omega_0 t} \right] \left[\sum_m a_m e^{im\omega_0 t} \right] \\
&= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \sum_n \sum_m a_n e^{in\omega_0 t} a_m e^{im\omega_0 t} \\
&= \sum_n \sum_m a_n a_m \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt\, e^{i(n+m)\omega_0 t} \\
&= \sum_n a_n a_{-n} \\
&= \sum_n |a_n|^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.39)
\end{aligned}$$

Put this into words and it says that the time-average power received is the sum of many terms, each one of which can be interpreted as the amount of power coming in *at that frequency* $n\omega_0$. The Fourier coefficients squared (absolute-squared really) are then proportional to the part of the power at a particular frequency. The "power spectrum."

Other Applications

In section 10.2 Fourier series will be used to solve partial differential equations, leading to equations such as Eq. (10.15).
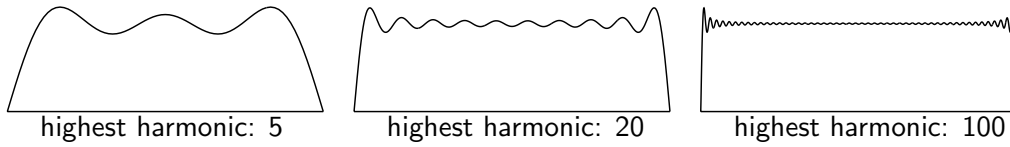
In quantum mechanics, Fourier series and its generalizations will manifest themselves in displaying the discrete energy levels of bound atomic and nuclear systems.

Music synthesizers are *all* about Fourier series and its generalizations.

5.7 Gibbs Phenomenon

There's a picture of the Gibbs phenomenon with Eq. (5.10). When a function has a discontinuity, its Fourier series representation will not handle it in a uniform way, and the series overshoots its goal at the discontinuity. The detailed calculation of this result is quite pretty, and it's an excuse to pull together several of the methods from the chapters on series and on complex algebra.

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\frac{(2k+1)\pi x}{L} = 1, \qquad (0 < x < L)$$

highest harmonic: 5   highest harmonic: 20   highest harmonic: 100

The analysis sounds straight-forward. Find the position of the first maximum. Evaluate the series there. It really is almost that clear. First however, you have to start with the a *finite* sum and find the first maximum of that. Stop the sum at $k = N$.

$$\frac{4}{\pi}\sum_{k=0}^{N}\frac{1}{2k+1}\sin\frac{(2k+1)\pi x}{L} = f_N(x) \qquad (5.40)$$

For a maximum, set the derivative to zero.

$$f_N'(x) = \frac{4}{L}\sum_{0}^{N}\cos\frac{(2k+1)\pi x}{L}$$

Write this as the real part of a complex exponential and use Eq. (2.3).

$$\sum_{0}^{N}e^{i(2k+1)\pi x/L} = \sum_{0}^{N}z^{2k+1} = z\sum_{0}^{N}z^{2k} = z\frac{1-z^{2N+2}}{1-z^2}$$

Factor these complex exponentials in order to put this into a nicer form.

$$= e^{i\pi x/L}\frac{e^{-i\pi x(N+1)/L}-e^{i\pi x(N+1)/L}}{e^{-i\pi x/L}-e^{i\pi x/L}}\frac{e^{i\pi x(N+1)/L}}{e^{i\pi x/L}} = \frac{\sin(N+1)\pi x/L}{\sin\pi x/L}e^{i\pi x(N+1)/L}$$

The real part of this changes the last exponential into a cosine. Now you have the product of the sine and cosine of $(N+1)\pi x/L$, and that lets you use the trigonometric double angle formula.

$$f_N'(x) = \frac{4}{L}\frac{\sin 2(N+1)\pi x/L}{2\sin\pi x/L} \qquad (5.41)$$

This is zero at the maximum. The first maximum after $x = 0$ is at $2(N+1)\pi x/L = \pi$, or $x = L/2(N+1)$.

Now for the value of f_N at this point,

$$f_N\big(L/2(N+1)\big) = \frac{4}{\pi}\sum_{k=0}^{N}\frac{1}{2k+1}\sin\frac{(2k+1)\pi L/2(N+1)}{L} = \frac{4}{\pi}\sum_{k=0}^{N}\frac{1}{2k+1}\sin\frac{(2k+1)\pi}{2(N+1)}$$

The final step is to take the limit as $N \to \infty$. As k varies over the set 0 to N, the argument of the sine varies from a little more than zero to a little less than π. As N grows you have the sum over a lot of terms, each of which is approaching zero. It's an integral. Let $t_k = k/N$ then $\Delta t_k = 1/N$. This sum is approximately

$$\frac{4}{\pi}\sum_{k=0}^{N}\frac{1}{2Nt_k}\sin t_k\pi = \frac{2}{\pi}\sum_{0}^{N}\Delta t_k\frac{1}{t_k}\sin t_k\pi \longrightarrow \frac{2}{\pi}\int_0^1\frac{dt}{t}\sin\pi t$$

In this limit $2k+1$ and $2k$ are the same, and $N+1$ is the same as N.

Finally, put this into a standard form by changing variables to $\pi t = x$.

$$\frac{2}{\pi}\int_0^{\pi}dx\,\frac{\sin x}{x} = \frac{2}{\pi}\,\mathrm{Si}(\pi) = 1.17898 \qquad\qquad \int_0^x dt\,\frac{\sin t}{t} = \mathrm{Si}(x) \qquad (5.42)$$

The function Si is called the "sine integral." It's just another tabulated function, along with erf, Γ, and others. This equation says that as you take the limit of the series, the first part of the graph approaches a vertical line starting from the origin, but it overshoots its target by 18%.

Exercises

1 A vector is given to be $\vec{A} = 5\hat{x} + 3\hat{y}$. Let a new basis be $\hat{e}_1 = (\hat{x} + \hat{y})/\sqrt{2}$, and $\hat{e}_2 = (\hat{x} - \hat{y})/\sqrt{2}$. Use scalar products to find the components of $\vec{A}$ in the new basis: $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2$.

2 For the same vector as the preceding problem, and another basis $\vec{f}_1 = 3\hat{x} + 4\hat{y}$ and $\vec{f}_2 = -8\hat{x} + 6\hat{y}$, express $\vec{A}$ in the new basis. Are these basis vectors orthogonal?

3 On the interval $0 < x < L$, sketch three graphs: the first term alone, then the second term alone, then the third. Try to get the scale of the graphs reasonable accurate. Now add the first two and graph. Then add the third also and graph. Do all this by hand, no graphing calculators, though if you want to use a calculator to calculate a few points, that's ok.

$$\sin\big(\pi x/L\big) - \tfrac{1}{9}\sin\big(3\pi x/L\big) + \tfrac{1}{25}\sin\big(5\pi x/L\big)$$

4 For what values of α are the vectors $\vec{A} = \alpha\hat{x} - 2\hat{y} + \hat{z}$ and $\vec{B} = 2\alpha\hat{x} + \alpha\hat{y} - 4\hat{z}$ orthogonal?

5 On the interval $0 < x < L$ with a scalar product defined as $\langle f, g\rangle = \int_0^L dx\, f(x)^*g(x)$, show that these are zero, making the functions orthogonal:

$$x \quad\text{and}\quad L - \tfrac{3}{2}x, \qquad \sin\pi x/L \quad\text{and}\quad \cos\pi x/L, \qquad \sin 3\pi x/L \quad\text{and}\quad L - 2x$$

6 Same as the preceding, show that these functions are orthogonal:

$$e^{2i\pi x/L} \quad \text{and} \quad e^{-2i\pi x/L}, \qquad L - \tfrac{1}{4}(7+i)x \quad \text{and} \quad L + \tfrac{3}{2}ix$$

7 With the same scalar product the last two exercises, for what values of α are the functions $f_1(x) = \alpha x - (1-\alpha)(L - \tfrac{3}{2}x)$ and $f_2(x) = 2\alpha x + (1+\alpha)(L - \tfrac{3}{2}x)$ orthogonal? What is the interpretation of the two roots?

8 Repeat the preceding exercise but use the scalar product $\langle f, g \rangle = \int_L^{2L} dx\, f(x)^* g(x)$.

9 Use the scalar product $\langle f, g \rangle = \int_{-1}^1 dx\, f(x)^* g(x)$, and show that the Legendre polynomials P_0, P_1, P_2, P_3 of Eq. (4.61) are mutually orthogonal.

10 Change the scalar product in the preceding exercise to $\langle f, g \rangle = \int_0^1 dx\, f(x)^* g(x)$ and determine if the same polynomials are still orthogonal.

11 Verify the examples stated on page 5.

Problems

5.1 Get the results in Eq. (5.18) by explicitly calculating the integrals.

5.2 (**a**) The functions with periodic boundary conditions, Eq. (5.20), are supposed to be orthogonal on $0 < x < L$. That is, $\langle u_n, u_m \rangle = 0$ for $n \neq m$. Verify this by explicit integration. What is the result if $n = m$ or $n = -m$? The notation is defined in Eq. (5.11). (**b**) Same calculation for the real version, $\langle u_n, u_m \rangle$, $\langle v_n, v_m \rangle$, and $\langle u_n, v_m \rangle$, Eq. (5.22)

5.3 Find the Fourier series for the function $f(x) = 1$ as in Eq. (5.10), but use as a basis the set of functions u_n on $0 < x < L$ that satisfy the differential equation $u'' = \lambda u$ with boundary conditions $u'(0) = 0$ and $u'(L) = 0$. (Eq. (5.23)) *Necessarily the first step* will be to examine all the solutions to the differential equation and to find the cases for which the bilinear concomitant vanishes.
(**b**) Graph the resulting Fourier series on $-2L < x < 2L$.
(**c**) Graph the Fourier series Eq. (5.10) on $-2L < x < 2L$.

5.4 (**a**) Compute the Fourier series for the function x^2 on the interval $0 < x < L$, using as a basis the functions with boundary conditions $u'(0) = 0$ and $u'(L) = 0$.
(**b**) Sketch the partial sums of the series for 1, 2, 3 terms. Also sketch this sum *outside* the original domain and see what this series produces for an extension of the original function. Ans: Eq. (5.1)

5.5 (**a**) Compute the Fourier series for the function x on the interval $0 < x < L$, using as a basis the functions with boundary conditions $u(0) = 0 = u(L)$. How does the coefficient of the nth term decrease as a function of n? (**b**) Also sketch this sum within *and* outside the original domain to see what this series produces for an extension of the original function.
Ans: $\frac{2L}{\pi} \sum_1^\infty \frac{(-1)^{n+1}}{n} \sin(n\pi x/L)$

5.6 (**a**) In the preceding problem the sine functions that you used don't match the qualitative behavior of the function x on this interval because the sine is zero at $x = L$ and x isn't. The qualitative behavior is different from the basis you are using for the expansion. You should be able to get better convergence for the series if you choose functions that more closely match the function that you're expanding, so try repeating the calculation using basis functions that satisfy $u(0) = 0$ and $u'(L) = 0$. How does the coefficient of the nth term decrease as a function of n? (**b**) As in the preceding problem, sketch some partial sums of the series and its extension outside the original domain. Ans: $\frac{8L}{\pi^2} \sum_0^\infty \left((-1)^n/(2n + 1)^2 \right) \sin \left((n + \frac{1}{2})\pi x/L \right)$

5.7 The function $\sin^2 x$ is periodic with period π. What is its Fourier series representation using as a basis functions that have this period? Eqs. (5.20) or (5.22).

5.8 In the two problems 5.5 and 5.6 you improved the convergence by choosing boundary conditions that better matched the function that you want. Can you do better? The function x vanishes at the origin, but its derivative isn't zero at L, so try boundary conditions $u(0) = 0$ and $u(L) = Lu'(L)$. These conditions match those of x so this ought to give even better convergence, but first you have to verify that these conditions guarantee the orthogonality of the basis functions. You have to verify that the left side of Eq. (5.15) is in fact zero. When you set up the basis, you will examine functions of the form $\sin kx$, but you will not be able to solve explicitly for the values of k. Don't worry about it. When you use Eq. (5.24) to get the coefficients all that you need to do is to use the equation that k satisfies to do the integrals. You do not need to have *solved* it. If you do all the algebra correctly you will probably have a surprise.

5.9 (a) Use the periodic boundary conditions on $-L < x < +L$ and basis $e^{\pi i n x/L}$ to write x^2 as a Fourier series. Sketch the sums up to a few terms. (b) Evaluate your result at $x = L$ where you know the answer to be L^2 and deduce from this the value of $\zeta(2)$.

5.10 On the interval $-\pi < x < \pi$, the function $f(x) = \cos x$. Expand this in a Fourier series defined by $u'' = \lambda u$ and $u(-\pi) = 0 = u(\pi)$. If you use your result for the series outside of this interval you define an extension of the original function. Graph this extension and compare it to what you normally think of as the graph of $\cos x$. As always, go back to the differential equation to get all the basis functions.
Ans: $-\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{2k+1}{(2k+3)(2k-1)} \sin\left((2k+1)(x+\pi)/2\right)$

5.11 Represent a function f on the interval $-L < x < L$ by a Fourier series using periodic boundary conditions

$$f(x) = \sum_{-\infty}^{\infty} a_n e^{n\pi i x/L}$$

(a) If the function f is odd, prove that for all n, $a_{-n} = -a_n$
(b) If the function f is even, prove that all $a_{-n} = a_n$.
(c) If the function f is real, prove that all $a_{-n} = a_n^*$.
(d) If the function is both real and even, characterize a_n.
(e) If the function is imaginary and odd, characterize a_n.

5.12 Derive the series Eq. (5.37).

5.13 For the function $e^{-\alpha t}$ on $0 < t < T$, express it as a Fourier series using periodic boundary conditions $[u(0) = u(T)$ and $u'(0) = u'(T)]$. Examine for plausibility the cases of large and small α. The basis functions for periodic boundary conditions can be expressed either as cosines and sines or as complex exponentials. Unless you can analyze the problem ahead of time and determine that it has some special symmetry that matches that of the trig functions, you're usually better off with the exponentials. Ans: $\left[(1 - e^{-\alpha T})/\alpha T\right]\left[1 + 2\sum_1^{\infty}[\alpha^2 \cos n\omega t + \alpha n\omega \sin n\omega t]/[\alpha^2 + n^2\omega^2]\right]$

5.14 (a) On the interval $0 < x < L$, write $x(L - x)$ as a Fourier series, using boundary conditions that the expansion functions vanish at the endpoints. Next, evaluate the series at $x = L/2$ to see if it gives an interesting result. (b) Finally, what does Parseval's identity tell you?
Ans: $\sum_1^{\infty} \frac{4L^2}{n^3\pi^3}\left[1 - (-1)^n\right]\sin(n\pi x/L)$

5.15 A full-wave rectifier takes as an input a sine wave, $\sin \omega t$ and creates the output $f(t) = |\sin \omega t|$. The period of the original wave is $2\pi/\omega$, so write the Fourier series for the output in terms of functions periodic with this period. Graph the function f first and use the graph to anticipate which terms in the Fourier series will be present.
When you're done, use the result to evaluate the infinite series $\sum_1^{\infty}(-1)^{k+1}/(4k^2 - 1)$
Ans: $\pi/4 - 1/2$

5.16 A half-wave rectifier takes as an input a sine wave, $\sin \omega t$ and creates the output

$$\sin \omega t \quad \text{if } \sin \omega t > 0 \qquad \text{and} \qquad 0 \quad \text{if } \sin \omega t \leq 0$$

The period of the original wave is $2\pi/\omega$, so write the Fourier series for the output in terms of functions periodic with this period. Graph the function first. Check that the result gives the correct value at $t = 0$, manipulating it into a telescoping series. Sketch a few terms of the whole series to see if it's

heading in the right direction.
Ans: $4/\pi + 1/2 \sin \omega t - 8/\pi \sum_{n \text{ even} >0} \cos(n\omega t)/(n^2 - 1)$

5.17 For the undamped harmonic oscillator, apply an oscillating force (cosine). This is a simpler version of Eq. (5.30). Solve this problem and add the general solution to the homogeneous equation. Solve this subject to the initial conditions that $x(0) = 0$ and $v_x(0) = v_0$.

5.18 The average (arithmetic mean) value of a function is

$$\langle f \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt\, f(t) \quad \text{or} \quad \langle f \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} dt\, f(t)$$

as appropriate for the problem.
What is $\langle \sin \omega t \rangle$? What is $\langle \sin^2 \omega t \rangle$? What is $\langle e^{-at^2} \rangle$?
What is $\langle \sin \omega_1 t \, \sin \omega_2 t \rangle$? What is $\langle e^{i\omega t} \rangle$?

5.19 In the calculation leading to Eq. (5.39) I assumed that $f(t)$ is real and then used the properties of a_n that followed from that fact. Instead, make no assumption about the reality of $f(t)$ and compute

$$\langle |f(t)|^2 \rangle = \langle f(t)^* f(t) \rangle$$

Show that it leads to the same result as before, $\sum |a_n|^2$.

5.20 The series

$$\sum_{n=0}^{\infty} a^n \cos n\theta \qquad (|a| < 1)$$

represents a function. Sum this series and determine what the function is. While you're about it, sum the similar series that has a sine instead of a cosine. Don't try to do these separately; combine them and do them as one problem. And check some limiting cases of course. And graph the functions.
Ans: $a \sin \theta / (1 + a^2 - 2a \cos \theta)$

5.21 Apply Parseval's theorem to the result of problem 5.9 and see what you can deduce.

5.22 If you take all the elements u_n of a basis and multiply each of them by 2, what happens to the result for the Fourier series for a given function?

5.23 In the section 5.3 several bases are mentioned. Sketch a few terms of each basis.

5.24 A function is specified on the interval $0 < t < T$ to be

$$f(t) = \begin{cases} 1 & (0 < t < t_0) \\ 0 & (t_0 < t < T) \end{cases} \qquad 0 < t_0 < T$$

On this interval, choose boundary conditions such that the left side of the basic identity (5.15) is zero. Use the corresponding choice of basis functions to write f as a Fourier series on this interval.

5.25 Show that the boundary conditions $u(0) = 0$ and $\alpha u(L) + \beta u'(L) = 0$ make the bilinear concomitant in Eq. (5.15) vanish. Are there any restrictions on α and β? Do not automatically assume that these numbers are real.

5.26 Derive a Fourier series for the function

$$f(x) = \begin{cases} Ax & (0 < x < L/2) \\ A(L - x) & (L/2 < x < L) \end{cases}$$

Choose the Fourier basis that you prefer. Evaluate the resulting series at $x = L/2$ to check the result. Sketch the sum of a couple of terms. Comment on the convergence properties of the result. Are they what you expect? What does Parseval's identity say?
Ans: $(2AL/\pi^2) \sum_{k \text{ odd}} (-1)^{(k-1)/2} \sin(k\pi x/L)/k^2$

5.27 Rearrange the solution Eq. (5.32) into a more easily understood form. (**a**) Write the first denominator as

$$-m\omega_e^2 + b\,i\omega_e + k = Re^{i\phi}$$

What are R and ϕ? The second term does not require you to repeat this calculation, just use its results, now combine everything and write the answer as an amplitude times a phase-shifted cosine.
(**b**) Assume that b is not too big and plot both R and ϕ versus the forcing frequency ω_e. Also, and perhaps more illuminating, plot $1/R$.

5.28 Find the form of Parseval's identity appropriate for power series. Assume a scalar product $\langle f, g \rangle = \int_{-1}^1 f(x)^* g(x) dx$ for the series $f(x) = \sum_0^\infty a_n x^n$, and $g(x) = \sum_0^\infty b_n x^n$, expressing the result in terms of matrices. Next, test your result on a simple, low-order polynomial.
Ans: $(a_0^*\ a_1^*\ldots)M(b_0\ b_1\ldots)^\sim$ where $M_{00} = 2$, $M_{02} = 2/3$ $M_{04} = 2/5$, ... and $\sim$ is transpose.

5.29 (**a**) In the Gibbs phenomenon, after the first maximum there is a first *minimum*. Where is it? how big is the function there? What is the limit of this point? That is, repeat the analysis of section 5.7 for this minimum point.
(**b**) While you're about it, what will you get for the limit of the sine integral, $\mathrm{Si}(\infty)$? The last result can also be derived by complex variable techniques of chapter 14, Eq. (14.16). Ans: $(2/\pi)\,\mathrm{Si}(2\pi) = 0.9028$

5.30 Make a blown-up copy of the graph preceding Eq. (5.40) and measure the size of the overshoot. Compare this experimental value to the theoretical limit. Same for the first minimum.

5.31 Find the power series representation about the origin for the sine integral, Si, that appeared in Eq. (5.42). What is its domain of convergence?
Ans: $\frac{2}{\pi} \sum_0^\infty (-1)^n \left(x^{2n+1}/(2n+1)(2n+1)!\right)$

5.32 An input potential in a circuit is given to be a square wave $\pm V_0$ at frequency ω. What is the voltage between the points a and b? In particular, assume that the resistance is small, and show that you can pick values of the capacitance and the inductance so that the output potential is almost exactly a sine wave at frequency 3ω. A filter circuit. Recall section 4.8.
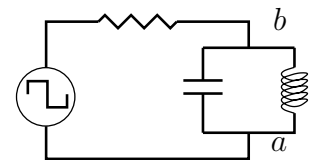
5.33 For the function $\sin(\pi x/L)$ on $(0 < x < 2L)$, expand it in a Fourier series using as a basis the trigonometric functions with the boundary conditions $u'(0) = 0 = u'(2L)$, the cosines. Graph the resulting series as extended outside the original domain.

5.34 For the function $\cos(\pi x/L)$ on $(0 < x < 2L)$, expand it in a Fourier series using as a basis the trigonometric functions with the boundary conditions $u(0) = 0 = u(2L)$, the sines. Graph the resulting series as extended outside the original domain.

5.35 (a) For the function $f(x) = x^4$, evaluate the Fourier series on the interval $-L < x < L$ using periodic boundary conditions $(u(-L) = u(L)$ and $u'(-L) = u'(L))$. (b) Evaluate the series at the point $x = L$ to derive the zeta function value $\zeta(4) = \pi^4/90$. Evaluate it at $x = 0$ to get a related series.

Ans: $\frac{1}{5}L^4 + L^4 \sum_1^\infty (-1)^n \left[\frac{8}{n^2\pi^2} - \frac{48}{n^4\pi^4}\right] \cos n\pi x/L$

5.36 Fourier series depends on the fact that the sines and cosines are orthogonal when integrated over a suitable interval. There are other functions that allow this too, and you've seen one such set. The Legendre polynomials that appeared in section 4.11 in the chapter on differential equations satisfied the equations (4.62). One of these is

$$\int_{-1}^1 dx\, P_n(x)P_m(x) = \frac{2}{2n+1}\delta_{nm}$$

This is an orthogonality relation, $\langle P_n, P_m \rangle = 2\delta_{nm}/(2n+1)$, much like that for trigonometric functions. Write a function $f(x) = \sum_0^\infty a_n P_n(x)$ and deduce an expression for evaluating the coefficients a_n. Apply this to the function $f(x) = x^2$.

5.37 For the standard differential equation $u'' = \lambda u$, use the boundary conditions $u(0) = 0$ and $2u(L) = Lu'(L)$. This is a special case of problem 5.25, so the bilinear concomitant vanishes. If you haven't done that problem, at least do this special case. Find all the solutions that satisfy these conditions and graph a few of them. You will not be able to find an explicit solution for the λs, but you can estimate a few of them to sketch graphs. Did you get them *all*?

5.38 Examine the function on $-L < x < L$ given by

$$f(x) = \begin{cases} 0 & (-L < x < -L/2) \text{ and } (L/2 < x < L) \\ 1 & (0 < x < L/2) \\ -1 & (-L/2 < x < 0) \end{cases}$$

Draw it first. Now find a Fourier series representation for it. You may choose to do this by doing lots of integrals, OR you may prefer to start with some previous results in this chapter, change periods, add or subtract, and do no integrals at all.

5.39 In Eq. (5.30) I wrote $\cos \omega_e t$ as the sum of two exponentials, $e^{i\omega_e t} + e^{-i\omega_e t}$. Instead, write the cosine as $e^{i\omega_e t}$ with the understanding that at the end you take the real part of the result, Show that the result is the same.

5.40 From Eq. (5.41) write an approximate closed form expression for the partial sum $f_N(x)$ for the region $x \ll L$ but *not* necessarily $x \ll NL$, though that extra-special case is worth doing too.

5.41 Evaluate the integral $\int_0^L dx\, x^2$ using the series Eq. (5.1) and using the series (5.2).

5.42 The Fourier series in problem 5.5 uses the same basis as the series Eq. (5.10). What is the result of evaluating the scalar products $\langle 1, 1 \rangle$ and $\langle 1, x \rangle$ with these series?

5.43 If you evaluated the $n = m$ case of Eq. (5.6) by using a different trig identity, you can do it by an alternative method: say that n and m in this equation aren't necessarily integers. Then take the limit as $n \to m$.